2016

Optimal Control and Its Application to the Life-Cycle Savings Problem

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Optimal Control and Its Application to the Life-Cycle Savings Problem

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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Richmond, Virginia
April 2016

Acknowledgements

I would like to give a huge thank you to my wonderful thesis advisor, Dr. Norma Ortiz-Robinson! You have dedicated so much of your time and effort to helping me explore applications of optimal control theory. You have stuck by my side throughout this entire process, and I couldn't have done it without you! I would also like to thank my thesis committee and the entire Department of Mathematical Sciences at Virginia Commonwealth University for supporting me and giving me the tools for success.

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Abstract

Throughout the course of this thesis, we give an introduction to optimal control theory and its necessary conditions, prove Pontryagin's Maximum Principle, and present the life-cycle saving under uncertain lifetime optimal control problem. We present a very involved sensitivity analysis that determines how a change in the initial wealth, discount factor, or relative risk aversion coefficient may affect the model the terminal depletion of wealth time, optimal consumption path, and optimal accumulation of wealth path. Through simulation of the life-cycle saving under uncertain lifetime model, we are not only able to present the model dynamics through time, but also to demonstrate the feasibility of the model.

Chapter 1

Introduction to Optimal Control and its Applications

When mathematically modeling a physical system, it is not uncommon to make an attempt to optimize some outcome by varying the parameters of that system. Thanks to the work of Lev Pontryagin and Richard Bellman in the 1950's, optimal control theory was born and quickly became a standard approach to finding a set of control strategies that will optimize the outcome of a physical system. Today, optimal control theory is a commonly used optimization technique - especially for the fields of engineering, physical sciences and economics.

The process of setting up an optimal control problem typically involves three main components:

1. A deterministic mathematical model that describes the evolution over time of the physical system that will be controlled,

2. A clear indicator of the performance of the model - e.g., Will you be minimizing or maximizing the outcome? In an economic model, you may wish to maximize profits or minimize costs,

3. A set of variable constraints that properly define the components of the model.

We will define the set of admissible controls as the set of control functions that abide by the constraints of the model. The optimal admissible controls will give the optimal performance of the model. Throughout the rest of the paper, optimal controls will be denoted with the asterisk symbol (*).

Let's now demonstrate an economic application of optimal control theory. Consider a small business that needs to produce 250 products to fill an order by time T. A reasonable business will strive to minimize the overall cost of filling the order. Consider also that the unit production costs will increase linearly with the rate of production and the cost to store each product will be held constant per unit time. Define $x(t)$ as the total number of products that have been produced by time t. By definition, $x(0) = 0$ and $x(T) = 250$. The time derivative of x will then represent the instantaneous rate of change of inventory, or the rate of production. The total cost C of production at time t will then be

$$C(t) = [c_1\dot{x}(t)]\,\dot{x}(t) + c_2x(t), \tag{1.1}$$

where c_1 and c_2 are constants. Notice that the first term in equation (1.1), $[c_1\dot{x}(t)]\dot{x}(t)$, represents the production costs and the second, $c_2x(t)$ represents the cost of holding inventory. The optimal control problem then becomes

$$\min_{\dot{x}} \int_0^T c_1(\dot{x}(t))^2 + c_2x(t)\,dt \tag{1.2}$$

$$x(0) = 0 \tag{1.3}$$

$$x(T) = 250 \tag{1.4}$$

$$\dot{x}(t) \geqslant 0, \tag{1.5}$$

where we are trying to find an optimal rate of production $\dot{x}^*(t)$ and an optimal accumulation of inventory $x^*(t)$. To minimize the total cost of production for $t \in [0, T]$, we may use optimal control techniques to manipulate our control variable $\dot{x}(t)$. Notice that the rate of production $\dot{x}(t)$ need not be continuous everywhere, but rather piecewise continuous.

Definition 1. *A function* $f : [a, b] \to \mathbb{R}$ *is piecewise continuous on an interval* $[a, b]$ *if and only if*

1. *There exists a partition on the interval* $[a, b]$ *such that* $a = x_0 < x_1 < \cdots < x_n = b$ *and* $x_1, x_2, \cdots, x_{n-1}$ *are discontinuities in the graph,*
2. $f(x)$ *is continuous on* (x_i, x_{i+1}) *for* $i = 0, 1, \cdots, n-1$,
3. $\lim_{x \to x_i^-} f(x) \neq +\infty, -\infty$, *and*
4. $\lim_{x \to x_i^+} f(x) \neq +\infty, -\infty$.

We explore a much more involved and in depth economic application to optimal control theory in Chapter 3, which relies heavily on the work completed by Siu Fai Leung [3–5].

Chapter 2

Necessary Conditions

2.1 Pontryagin Maximum Principle

Theorem 2.1.1 (Pontryagin's Maximum Principle). *Suppose that* $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *are continuously differentiable. Let* $\mathscr{C}[t_0, t_1]$ *be the set of all continuous functions with domain* $[t_0, t_1]$*. Now consider the optimization problem*

$$\max_{\mathbf{u} \in \mathbb{U}} \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t)) dt, \tag{2.1}$$

subject to the following constraints:

$$\begin{cases} \dot{\mathbf{x}}(t) = g(t, \mathbf{x}(t), \mathbf{u}(t)) & i = 1, \cdots, n \\ \mathbf{x}(t_0) = \boldsymbol{\alpha} \\ \mathbb{U} = \{\mathbf{u} : [t_0, t_1] \to \mathbb{R}^k, \mathbf{u} \in \mathscr{C}([t_0, t_1])\} \end{cases} \tag{2.2}$$

where $\mathbb{U}$ *is the set of all admissible controls, is non empty, and is open. Define* $\mathbf{u^*}$ *as the set of optimal admissible controls and* $\mathbf{x^*}$ *the associated optimal trajectory. Then there exists a continuous*

Lagrange multiplier $\boldsymbol{\lambda}^* : [t_0, t_1] \to \mathbb{R}^n$ *such that*

$$\nabla_{\mathbf{u}} H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t)) = \mathbf{0}, \qquad \forall t \in [t_0, t_1], \tag{2.3}$$

$$\nabla_{\mathbf{x}} H(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t)) = -\dot{\boldsymbol{\lambda}}^*(t), \qquad \forall t \in [t_0, t_1], \tag{2.4}$$

$$\boldsymbol{\lambda}^*(t_1) = 0, \tag{2.5}$$

where the Hamiltonian function H *is defined to be*

$$H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) = f(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda} \cdot g(t, \mathbf{x}, \mathbf{u}) \tag{2.6}$$

[1].

We will prove in detail the Pontryagin Maximum Principle for $n = 3$ under the regularity assumptions stated in Theorem 2.1.1. Our proof relies on the following two lemmas:

Lemma 2.1.2. *Suppose* $f(x) \in \mathscr{C}[a, b]$ *and* $\int_a^b f(x) \cdot g(x) dx = 0 \ \forall g(x) \in \mathscr{C}[a, b]$. *Then* $f(x) = 0$ *on the entire interval* $[a, b]$.

Lemma 2.1.3. *(Leibniz). Suppose* $F : [a, b] \times \mathbb{R}^3 \to \mathbb{R}^3$ *is continuous and* $\nabla_h F(t, h)$ *exists and is continuous in* (t, h). *Then* $\int_a^b F(t, h) dt$ *is differentiable and* $\frac{d}{dh} \int_a^b F(t, h) dt = \int_a^b \frac{\partial}{\partial h} F(t, h) dt$ *[8].*

The proof of the Pontryagin Maximum Principle for $n = 3$ is as follows.

Proof. Let $J(\mathbf{u}) = \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t)) dt$. Define $\mathbf{u}^* \in \mathscr{C}$ with $k = 3$ to be an optimal control and $\mathbf{x}^*$ to be the associated trajectory. Let us fix a continuous function $\mathbf{h} = (h_1, h_2, h_3) : [t_0, t_1] \to \mathbb{R}^3$. For every displacement constant $\boldsymbol{\epsilon} \in \mathbb{R}^3$ we define the function $\mathbf{u}_{\boldsymbol{\epsilon}} : [\mathbf{t_0}, \mathbf{t_1}] \to \mathbb{R}^3$ as

$$\mathbf{u}_{\boldsymbol{\epsilon}} = \mathbf{u}^* + \boldsymbol{\epsilon} \cdot \mathbf{h} = (u_1^* + \epsilon_1 h_1, u_2^* + \epsilon_2 h_2, u_3^* + \epsilon_3 h_3). \tag{2.7}$$

Since $\mathbb{U}$ is a nonempty and open set of continuous admissible controls, we can say that there exists an $\boldsymbol{\epsilon}$ with $||\boldsymbol{\epsilon}||$ sufficiently small, $\mathbf{u}_{\boldsymbol{\epsilon}}$ is an admissible control. We denote $\mathbf{x}_{\boldsymbol{\epsilon}} : [t_0, t_1] \to \mathbb{R}^3$ to be the associated trajectory for $\mathbf{u}_{\boldsymbol{\epsilon}}$. Then we can define a function $\mathscr{J}_{\mathbf{h}} : \mathbb{R}^3 \to \mathbb{R}$ as

$$\mathscr{J}_{\mathbf{h}}(\boldsymbol{\epsilon}) = \int_{t_0}^{t_1} f(t, \mathbf{x}_{\boldsymbol{\epsilon}}(t), \mathbf{u}_{\boldsymbol{\epsilon}}(t))dt \tag{2.8}$$

It is clear that when $\boldsymbol{\epsilon} = \mathbf{0}$, $\mathbf{u}_{\mathbf{0}}(t) = \mathbf{u}^*(t)$, $\mathbf{x}_{\mathbf{0}}(t) = \mathbf{x}^*(t)$, and $\mathbf{x}_{\boldsymbol{\epsilon}}(t_0) = \boldsymbol{\alpha}$. Then

$$\mathscr{J}_{\mathbf{h}}(\mathbf{0}) = \int_{t_0}^{t_1} f(t, \mathbf{x}^*(t), \mathbf{u}^*(t))dt. \tag{2.9}$$

Therefore, since the control set, $\mathbb{U}$, is convex and $\mathbf{u}^*$ is optimal, $\mathscr{J}_{\mathbf{h}}(\mathbf{0}) \geqslant \mathscr{J}_{\mathbf{h}}(\boldsymbol{\epsilon})$ for all $\boldsymbol{\epsilon}$. Then $\mathscr{J}_{\mathbf{h}}$ has a local maximum at $\boldsymbol{\epsilon} = \mathbf{0}$. Thus, $\nabla_{\boldsymbol{\epsilon}} \mathscr{J}_{\mathbf{h}}(\mathbf{0}) = \mathbf{0}$. Let $\boldsymbol{\lambda} : [t_0, t_1] \to \mathbb{R}^3$ be a continuous function. Recall the constraint, $\dot{\mathbf{x}}(t) = g(t, \mathbf{x}(t), \mathbf{u}(t))$, from our original optimization problem defined in the statement of the theorem. Then we have

$$\begin{aligned}
\mathscr{J}_{\mathbf{h}}(\boldsymbol{\epsilon}) &= \int_{t_0}^{t_1} f(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}})dt \\
&= \int_{t_0}^{t_1} [f(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}) + \boldsymbol{\lambda}(g(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}) - \dot{\mathbf{x}}_{\boldsymbol{\epsilon}})dt] \\
&= \int_{t_0}^{t_1} [H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) - \boldsymbol{\lambda}\dot{\mathbf{x}}_{\boldsymbol{\epsilon}}]dt \\
&= \int_{t_0}^{t_1} H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda})dt - \int_{t_0}^{t_1} \boldsymbol{\lambda}\dot{\mathbf{x}}_{\boldsymbol{\epsilon}}dt.
\end{aligned} \tag{2.10}$$

We can integrate the right most integral by using integration by parts:

$$\begin{aligned}
\mathscr{J}_{\mathbf{h}}(\boldsymbol{\epsilon}) &= \int_{t_0}^{t_1} H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda})dt - [(\boldsymbol{\lambda}\mathbf{x}_{\boldsymbol{\epsilon}}|_{t_0}^{t_1} - \int_{t_0}^{t_1} \mathbf{x}_{\boldsymbol{\epsilon}}\dot{\boldsymbol{\lambda}}dt] \\
&= \int_{t_0}^{t_1} [H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) + \mathbf{x}_{\boldsymbol{\epsilon}}\dot{\boldsymbol{\lambda}}]dt - \left(\boldsymbol{\lambda}\mathbf{x}_{\boldsymbol{\epsilon}}|_{t_0}^{t_1}\right) \\
&= \int_{t_0}^{t_1} \left[H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) + \mathbf{x}_{\boldsymbol{\epsilon}}\dot{\boldsymbol{\lambda}}\right] dt - \boldsymbol{\lambda}(t_1)\mathbf{x}_{\boldsymbol{\epsilon}}(t_1) + \boldsymbol{\lambda}(t_0)\mathbf{x}_{\boldsymbol{\epsilon}}(t_0).
\end{aligned} \tag{2.11}$$

Then by taking the partial derivative of $\mathscr{J}_{\mathbf{h}}(\boldsymbol{\epsilon})$ with respect to ϵ_i for $1 \leqslant i \leqslant 3$, we have

$$\frac{\partial \mathscr{J}_{\mathbf{h}}(\boldsymbol{\epsilon})}{\partial \epsilon_i} = \int_{t_0}^{t_1} [\nabla_x H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) \cdot \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t) + \nabla_u H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) \cdot \nabla_{\epsilon_i} \mathbf{u}_{\boldsymbol{\epsilon}}(t) + \\ + \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}} \dot{\boldsymbol{\lambda}}] dt - \boldsymbol{\lambda}(t_1) \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t_1) + \boldsymbol{\lambda}(t_0) \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t_0). \quad (2.12)$$

Since we know that $\mathbf{u}_{\boldsymbol{\epsilon}} = (u_1^* + \epsilon_1 h_1, u_2^* + \epsilon_2 h_2, u_3^* + \epsilon_3 h_3)$, we can see that

$$\nabla_{\epsilon_1} \mathbf{u}_{\boldsymbol{\epsilon}} = (h_1, 0, 0)$$
$$\nabla_{\epsilon_2} \mathbf{u}_{\boldsymbol{\epsilon}} = (0, h_2, 0)$$
$$\nabla_{\epsilon_3} \mathbf{u}_{\boldsymbol{\epsilon}} = (0, 0, h_3).$$

Then we have

$$\frac{\partial \mathscr{J}_{\mathbf{h}}(\boldsymbol{\epsilon})}{\partial \epsilon_i} = \int_{t_0}^{t_1} [\nabla_x H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) \cdot \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t) + \frac{\partial}{\partial u_i} H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) \cdot h_i(t) + \\ + \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}} \dot{\boldsymbol{\lambda}}] dt - \boldsymbol{\lambda}(t_1) \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t_1) + \boldsymbol{\lambda}(t_0) \nabla_{\epsilon_i} \boldsymbol{\alpha} \\ = \int_{t_0}^{t_1} [\nabla_x H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) \cdot \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t) + \frac{\partial}{\partial u_i} H(t, \mathbf{x}_{\boldsymbol{\epsilon}}, \mathbf{u}_{\boldsymbol{\epsilon}}, \boldsymbol{\lambda}) \cdot h_i(t) + \\ + \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}} \dot{\boldsymbol{\lambda}}] dt - \boldsymbol{\lambda}(t_1) \nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t_1). \quad (2.13)$$

At $\boldsymbol{\epsilon} = \mathbf{0}$, this is

$$\frac{\partial \mathscr{J}_{\mathbf{h}}(\mathbf{0})}{\partial \epsilon_i} = \int_{t_0}^{t_1} [\nabla_x H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}) \cdot (\nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t)|_{\boldsymbol{\epsilon}=0}) + \frac{\partial}{\partial u_i} H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}) \cdot h_i(t) + \\ + \dot{\boldsymbol{\lambda}} (\nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}|_{\boldsymbol{\epsilon}=0})] dt - \boldsymbol{\lambda}(t_1) (\nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t_1)|_{\boldsymbol{\epsilon}=0}). \quad (2.14)$$

Define $\boldsymbol{\lambda}$ so that $\dot{\boldsymbol{\lambda}} = -\nabla_x H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda})$ for $t \in [t_0, t_1]$ and $\boldsymbol{\lambda}(t_1) = 0$. Since we know that

$$\nabla_x H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}) = \nabla_x f(t, \mathbf{x}^*, \mathbf{u}^*) + \boldsymbol{\lambda} \cdot \nabla_x f(t, \mathbf{x}^*, \mathbf{u}^*), \quad (2.15)$$

we also know that $\dot{\boldsymbol{\lambda}} = -\nabla_x H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda})$ is linear in $\boldsymbol{\lambda}$ and f has continuous first partial derivatives. Therefore, a unique solution $\boldsymbol{\lambda}^*$ exists. Now let $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$. Then (2.14) becomes

$$\begin{aligned}\frac{\mathscr{J}_{\mathbf{h}}(\mathbf{0})}{\partial \epsilon_i} &= \int_{t_0}^{t_1} \Big[\Big(\nabla_x H(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*) + \dot{\boldsymbol{\lambda}}^*\Big) \cdot (\nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}|_{\epsilon=0}) \\ &\qquad + \left[\frac{\partial H}{\partial u_i}(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}) h_i(t)\right] dt - \boldsymbol{\lambda}^*(t_1) \left(\nabla_{\epsilon_i} \mathbf{x}_{\boldsymbol{\epsilon}}(t_1)|_{\epsilon=0}\right) \\ &= \int_{t_0}^{t_1} \frac{\partial H}{\partial u_i}(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*) h_i(t) dt. \end{aligned} \tag{2.16}$$

Since $\mathscr{J}_h$ attains a maximum at $\boldsymbol{\epsilon} = \mathbf{0}$, $\nabla_{\boldsymbol{\epsilon}} \mathscr{J}_h(\mathbf{0}) = \mathbf{0}$ is defined to be equal to $\mathbf{0}$, we have

$$\mathbf{0} = \int_{t_0}^{t_1} \frac{\partial H}{\partial u_i}(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*) h_i(t) dt. \tag{2.17}$$

for $i \in [1, 3]$. Therefore, $\frac{\partial H}{\partial u_i}(t, \mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*) = 0$, proving Theorem 2.1.1 for $n = k = 3$ [1]. □

Since we will be optimizing a control problem with state constraints in Chapter 3, we will require the use of the following maximum principle for optimal control problems with state constraints:

Theorem 2.1.4. *Consider the following optimal control problem with state constraints:*

$$\max \mathbf{u} \in \mathbb{U} \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t)) dt \tag{2.18}$$

$$\dot{\mathbf{x}}(t) = g(t, \mathbf{x}(t), \mathbf{u}(t)), \qquad \mathbf{x}(t_0) = \boldsymbol{\alpha}, \tag{2.19}$$

$$\mathbf{u}(t) \geqslant 0, \tag{2.20}$$

$$\mathbf{x}(t) \geqslant 0, \tag{2.21}$$

where f, g are piecewise continuously differentiable. Let $\mathbf{u}^*$ be an optimal control and $\mathbf{x}^*$ the associated optimal trajectory over the interval $[t_0, t_1]$. Suppose that $\mathbf{u}^*$ is right-continuous with left-hand limits and satisfies the constraints defined above. Define the Lagrangian function to be *the following:*

$$L(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(t, \mathbf{x}(t), \mathbf{u}(t)) + \boldsymbol{\lambda} \cdot g(t, \mathbf{x}(t), \mathbf{u}(t)) + \mu_1 g(t, \mathbf{x}(t), \mathbf{u}(t)) + \mu_2 h(\mathbf{x}(t), t). \quad (2.22)$$

Then there exists a piecewise absolutely continuous function $\boldsymbol{\lambda} : [t_0, t_1] \to \mathbb{R}$ *and piecewise continuous functions* $\mu_1, \mu_2 : [t_0, t_1] \to \mathbb{R}$ *such that*

$$\nabla_{\mathbf{u}} L(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}) = \mathbf{0}, \quad (2.23)$$

$$\nabla_{\mathbf{x}} L(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}) = -\dot{\boldsymbol{\lambda}}^*, \quad (2.24)$$

$$\mu_1(t) \geqslant 0, \qquad \mu_1(t) g^* = \mathbf{0}, \quad (2.25)$$

$$\mu_2(t) \geqslant 0, \qquad \mu_2(t) h^* = \mathbf{0} \quad (2.26)$$

[2].

Definition 2. *A function* f *is said to be absolutely continuous on the interval* $[a, b]$ *if* f *is defined on* $[a, b]$, f' *exists almost everywhere,* f *is Lebesgue integrable on* $[a, b]$, *and* $f(x) = f(a) + \int_a^x f'(t)dt$ *for* $x \in [a, b]$ *[6].*

Chapter 3

Life-Cycle Saving Under Uncertain Lifetime Model

3.1 Model Development and Optimal Solutions

Suppose that a consumer expects to have a lifetime of T years and is working to find an optimal consumption plan that will allow them to earn and distribute wealth through savings and consumption. Suppose that this consumer begins with no wealth and wishes to use up all accumulated wealth by the time of his or her death. The preferences of the consumer may be given by the following Fisher utility function:

$$V(c) = \int_0^T \alpha(t) g[c(t)] dt, \tag{3.1}$$

where $\alpha(t)$ is a discount function, $c(t)$ is a consumption plan, and $g(c)$ is the utility function that is associated with the rate of consumption throughout time. Before going any further, we will make the following assumptions about the attributes of the Fisher utility function:

- $c(t)$ is piecewise continuous on $[0, T]$

- $\alpha(t)$ is positive and continuously differentiable on $[0, T]$

- $g(c)$ is a concave, continuously differentiable function on $(0, \infty)$, where $g'(c) > 0$ and $g''(c) < 0$.

We can also define the consumer's net wealth at time t as the following:

$$S(t) = \int_0^t \left[\left(e^{\int_\tau^t j(x)dx} \right) (m(\tau) - c(\tau)) \right] d\tau, \tag{3.2}$$

where $j(\tau)$ is the interest rate that begins at time τ and $m(\tau)$ is the income function. Assuming that $j(\tau)$ is non-negative and continuous on $[0, T]$, and $m(\tau)$ is non-negative and piecewise continuously differentiable on $[0, T]$, then $S(t)$ will be a piecewise continuously differentiable function on $[0, T]$. $S(t)$ represents the net accumulation of wealth with an interest rate that is compounded continuously throughout time. In order to prevent having a negative amount of accumulated wealth at the time of death, we will define the wealth constraint

$$S(T) \geqslant 0 \tag{3.3}$$

[9].

Now suppose that the consumer has an uncertain lifetime with a maximum possible lifetime length of $\bar{T}$. Thus, we will define T as a random variable with a probability density function π on the interval $[0, \bar{T}]$. Then $\int_0^{\bar{T}} \pi(t)dt = 1$ and $\pi(t) > 0$ for $0 < t < \bar{T}$. The probability of a consumer surviving past time t can then be defined as

$$\Omega(t) = \int_t^{\bar{T}} \pi(\tau)d\tau = e^{-\int_0^t \pi_x(x)dx}, \qquad 0 \leqslant t \leqslant \bar{T}, \tag{3.4}$$

where $\pi_t(t) = \pi(t)/\Omega(t)$ is the mortality hazard function. The expected value of the consumer's preferences, represented by the Fisher utility function, is simply

$$\begin{aligned}\bar{V}(c) = \mathbb{E}[V(c)] &= \int_0^{\bar{T}} \pi(\tau) \int_0^{\tau} \alpha(t) g[c(t)]\, dt\, d\tau \\ &= \int_0^{\bar{T}} \alpha(t) g[c(t)] \int_t^{\bar{T}} \pi(\tau)\, d\tau\, dt \end{aligned} \tag{3.5}$$

Recalling that $\Omega(t) = \int_t^{\bar{T}} \pi(\tau)\, d\tau$, we can rewrite the expected utility for consumption plan c in the following way:

$$\bar{V}(c) = \int_0^{\bar{T}} \Omega(t)\alpha(t)g[c(t)]dt. \tag{3.6}$$

Our goal is to maximize the utility function $\bar{V}(c)$ with uncertain lifetime. To do so, we must find an optimal consumption plan $c(t) \in \Phi$, where Φ is the set of admissible controls. The problem then becomes:

$$\max_{c(t)\in\Phi} \bar{V}(c) = \max_{c(t)\in\Phi} \int_0^{\bar{T}} \Omega(t)\alpha(t)g[c(t)]dt \tag{3.7}$$

such that

$$c(t) \geqslant 0 \tag{3.8}$$

$$S(t) \geqslant 0 \tag{3.9}$$

$$S'(t) = j(t)S(t) + m(t) - c(t) \tag{3.10}$$

$$S(0) = S_0 \tag{3.11}$$

$$S(\bar{T}) = 0 \tag{3.12}$$

[4].

Before we can solve for the optimal consumption plan c^*, we must begin by defining the Hamiltonian function

$$H(c, S, t, \lambda) = \Omega(t)\alpha(t)g[c(t)] + \lambda(t)[j(t)S(t) + m(t) - c(t)] \tag{3.13}$$

and the Lagrangian Function

$$\begin{aligned} L(c, S, t, \lambda, \mu_1, \mu_2) =& \Omega(t)\alpha(t)g[c(t)] + \lambda(t)[j(t)S(t) + m(t) - c(t)] \\ &+ \mu_1(t)c(t) + \mu_2(t)S(t). \end{aligned} \tag{3.14}$$

From the necessary conditions of optimality, we have

$$\frac{\partial L(c^*(t), S^*(t), t, \mu_1, \mu_2)}{\partial c(t)} = \Omega(t)\alpha(t)g'[c^*(t)] - \lambda(t) + \mu_1(t) = 0 \tag{3.15}$$

$$\frac{\partial L(c^*(t), S^*(t), t, \mu_1, \mu_2)}{\partial S(t)} = \lambda(t)j(t) + \mu_2(t) = -\lambda'(t) \tag{3.16}$$

$$\mu_1(t) \geqslant 0, \mu_1(t)c^*(t) = 0 \tag{3.17}$$

$$\mu_2(t) \geqslant 0, \mu_2(t)S^*(t) = 0. \tag{3.18}$$

Through the use of an integrating factor, we can solve for $\lambda(t)$ in equation (3.16) above. Define the integrating factor to be $e^{\int_0^t j(x)dx}$. Then the solution to (3.16) is obtained as follows:

$$\begin{aligned} \lambda'(t) + \lambda(t)j(t) &= -\mu_2(t) \\ e^{\int_0^t j(x)dx}\lambda'(t) + e^{\int_0^t j(x)dx}\lambda(t) &= -e^{\int_0^t j(x)dx}\mu_2(t) \\ \int_0^t \frac{d}{dw}\left[e^{\int_0^w j(x)dx}\lambda(w)\right] dw &= -\int_0^t e^{\int_0^w j(x)dx}\mu_2(w)dw \\ e^{\int_0^t j(x)dx}\lambda(t) - e^{\int_0^0 j(x)dx}\lambda(0) &= -\int_0^t e^{\int_0^w j(x)dx}\mu_2(w)dw \\ e^{\int_0^t j(x)dx}\lambda(t) &= \lambda(0) - \int_0^t e^{\int_0^w j(x)dx}\mu_2(w)dw. \end{aligned} \tag{3.19}$$

Solving equation (3.19) for $\lambda(t)$ gives us

$$\lambda(t) = \lambda(0)e^{-\int_0^t j(x)dx} - e^{-\int_0^t j(x)dx}\int_0^t e^{\int_0^w j(x)dx}\mu_2(w)dw$$
$$\lambda(t) = \lambda(0)e^{-\int_0^t j(x)dx} - \int_0^t e^{-\int_w^t j(x)dx}\mu_2(w)dw. \tag{3.20}$$

Now that we have solved for $\lambda(t)$, we can substitute equation (3.20) into the necessary condition defined in equation (3.15):

$$\Omega(t)\alpha(t)g'[c^*(t)] - \lambda(t) + \mu_1(t) = 0$$

$$\Omega(t)\alpha(t)g'[c^*(t)] - \lambda(0)e^{-\int_0^t j(x)dx} + \int_0^t e^{-\int_w^t j(x)dx}\mu_2(w)dw + \mu_1(t) = 0. \tag{3.21}$$

Solving equation (3.21) for $\Omega(t)\alpha(t)g'[c^*(t)]$ then gives us

$$\Omega(t)\alpha(t)g'[c^*(t)] = \lambda(0)e^{-\int_0^t j(x)dx} - \int_0^t e^{-\int_w^t j(x)dx}\mu_2(w)dw - \mu_1(t). \tag{3.22}$$

We will use the following proposition to continue

Proposition 1. *Assume (3.8)-(3.12). If either* $\lim_{c\to 0^+} g'(c) < \infty$ *or* $m(\bar{T}) > 0$*, then there exists a* $t^* \in [0, \bar{T})$ *such that* $t^* = \min\{t \in [0, T) : S^*(z) = 0$ *and* $c^*(z) = m(z)$ *for all* $z \in [t, \bar{T}]\}$.

For this problem, we will assume that after retirement, and individual will have a constant income from a retirement annuity. Due to this assumption, we see that $m(\bar{T}) > 0$, and we will apply Proposition 1. Let $t = t^*$. Then by Proposition 1, $c^*(t^*) = m(t^*)$ and (3.22) becomes

$$\Omega(t^*)\alpha(t^*)g'[m(t^*)] = \lambda(0)e^{-\int_0^{t^*} j(x)dx} - \int_0^{t^*} e^{-\int_w^{t^*} j(x)dx}\mu_2(w)dw. \tag{3.23}$$

By solving equation (3.23) above for $\lambda(0)$, we have

$$\begin{aligned}\lambda(0) =& e^{\int_0^{t^*} j(x)\,dx}\Omega(t^*)\alpha(t^*)g'[m(t^*)] + e^{\int_0^{t^*} j(x)\,dx}\int_0^{t^*} e^{-\int_w^{t^*} j(x)\,dx}\mu_2(w)\,dw \\ =& e^{\int_0^{t^*} j(x)\,dx}\Omega(t^*)\alpha(t^*)g'[m(t^*)] + \int_0^{t^*} e^{\int_0^{w} j(x)\,dx}\mu_2(w)\,dw. \end{aligned} \tag{3.24}$$

If we now substitute equation (3.24) in for $\lambda(0)$ in (3.22), for $t \in [0, t^*]$ we have

$$\begin{aligned}\Omega(t)\alpha(t)g'[c^*(t)] =& e^{-\int_0^{t^*} j(x)\,dx}\left[e^{\int_0^{t^*} j(x)\,dx}\Omega(t^*)\alpha(t^*)g'[m(t^*)] + \int_0^{t^*} e^{\int_0^{w} j(x)\,dx}\mu_2(w)\,dw\right] \\ & - \int_0^{t} e^{-\int_w^{t} j(x)\,dx}\mu_2(w)\,dw - \mu_1(t) \\ =& e^{\int_t^{t^*} j(x)\,dx}\Omega(t^*)\alpha(t^*)g'[m(t^*)] + \int_0^{t^*} e^{-\int_w^{t} j(X)\,dx}\mu_2(w)\,dw \\ & - \int_0^{t} e^{-\int_w^{t} j(x)\,dx}\mu_2(w)\,dw - \mu_1(t). \end{aligned}$$

Thus,

$$\Omega(t)\alpha(t)g'[c^*(t)] = e^{\int_t^{t^*} j(x)\,dx}\Omega(t^*)\alpha(t^*)g'[m(t^*)] + \int_t^{t^*} e^{-\int_w^{t} j(x)\,dx}\mu_2(w)\,dw - \mu_1(t) \tag{3.25}$$

We can now use (3.25) to solve for the optimal consumption path $c^*(t)$:

$$c^*(t) = \begin{cases} g'^{-1}\left(\frac{e^{\int_t^{t^*} j(x)\,dx}\Omega(t^*)\alpha(t^*)g'[m(t^*)] + \int_t^{t^*} e^{-\int_w^{t} j(x)\,dx}\mu_2(w)\,dw - \mu_1(t)}{\Omega(t)\alpha(t)}\right), & t \in [0, t^*] \\ m(t), & t \in [t^*, \bar{T}]. \end{cases} \tag{3.26}$$

Since the accumulation of wealth function, $S(t)$, is also a controllable function that impacts the overall utility function, we will work to solve for the optimal accumulated savings path $S^*(t)$ that is associated with the optimal consumption path $c^*(t)$. To do this, we will begin by solving the ordinary differential equation from (3.11):

$$
\begin{aligned}
S'(t) &= j(t)S(t) + m(t) - c(t) \\
S'(t) - j(t)S(t) &= m(t) - c(t) \\
e^{-\int_0^t j(x)dx} S'(t) - j(t) e^{-\int_0^t j(x)dx} S(t) &= e^{-\int_0^t j(x)dx} \left(m(t) - c(t)\right) \\
\int_0^t \frac{d}{dz}\left[e^{-\int_0^z j(x)dx} S(z)\right] dz &= \int_0^t e^{-\int_0^z j(x)dx} \left(m(z) - c(z)\right) dz \\
e^{-\int_0^t j(X)dx} S(t) - S_0 &= \int_0^t e^{-\int_0^z j(x)dx} \left(m(z) - c(z)\right) dz. \qquad (3.27)
\end{aligned}
$$

Through simplification, we can solve (3.27) for the wealth function $S(t)$:

$$
S(t) = e^{\int_0^t j(x)dx}\left[S_0 + \int_0^t e^{-\int_0^z j(x)dx}\left(m(z) - c(z)\right) dz\right]. \qquad (3.28)
$$

Now let $t = t^*$, where t^* is the time at which all wealth of the consumer is depleted. Then $S(t^*) = 0$ and

$$
0 = e^{\int_0^{t^*} j(x)dx}\left[S_0 + \int_0^{t^*} e^{-\int_0^t j(x)dx}(m(t) - c(t))dt\right] \qquad (3.29)
$$

Note that $e^{\int_0^{t^*} j(x)dx} \neq 0$, and by rearranging equation (3.29), we have the following significant equality:

$$
\int_0^{t^*} e^{-\int_0^t j(x)dx} c(t)dt = S_0 + \int_0^{t^*} e^{-\int_0^t j(x)dx} m(t)dt. \qquad (3.30)
$$

The equality shown in (3.30) signifies that the total consumption until all wealth is depleted at time $t = t^*$ must be equal to the initial wealth plus the total income up until that particular point in time. We may also note that at time $t = t^*$, the initial wealth is simply

$$
S_0 = \int_0^{t^*} e^{-\int_0^t j(x)dx}(c(t) - m(t))dt. \qquad (3.31)
$$

The *optimal* accumulation of wealth function $S^*(t)$ is simply the accumulation of wealth function that is associated with the optimal consumption path $c^*(t)$:

$$S^*(t) = \begin{cases} e^{\int_0^t j(x)dx}\left[S_0 + \int_0^t e^{-\int_0^z j(x)dx}(m(z) - c^*(z))dz\right], & t \in [0, t^*] \\ 0, & t \in [t^*, \bar{T}]. \end{cases} \tag{3.32}$$

From this, we can say that once the total accumulation of wealth has been depleted at t^*, the accumulation of wealth will remain zero for the remainder of the lifetime of the consumer. From equations (3.26) and (3.31), the optimal terminal wealth depletion time t^* can be found with the following equation for S_0:

$$S_0 = \int_0^{t^*} e^{-\int_0^t j(x)dx} \left[(g')^{-1}\left(\frac{e^{\int_t^{t^*} j(x)dx}\Omega(t^*)\alpha(t^*)g'[m(t^*)] + \int_t^{t^*} e^{-\int_w^t j(x)dx}\mu_2(w)dw - \mu_1(t)}{\Omega(t)\alpha(t)}\right) - m(t)\right] dt \tag{3.33}$$

[5].

3.2 Sensitivity Analysis

To further expand on the dynamics of the life-cycle savings under uncertain lifetime model, we should investigate how a sudden increase or decrease in the initial wealth and discount parameters will affect the consumer's overall optimal wealth, consumption, and time until total wealth is depleted. To do this, we must first make the following simplifying assumptions that will allow us to properly and efficiently perform the analysis:

$$\alpha(t) = \exp(-\alpha t), \qquad \text{for } \alpha \geqslant 0 \tag{3.34}$$

$$j(t) = j \tag{3.35}$$

$$\mu_1 = \mu_2 = 0, \tag{3.36}$$

where α and j are real valued constants. Note that the simplifying assumptions made will not affect the results of the sensitivity analysis. With the assumptions, the optimal consumption, initial wealth, and optimal accumulated wealth functions on $t \in [0, t^*]$ are simplified to

$$\begin{aligned} c^*(t) &= (g')^{-1}\left(\frac{e^{\int_t^{t^*} j\,dx}\Omega(t^*)e^{-\alpha t^*}g'(m(t^*))}{\Omega(t)e^{-\alpha t}}\right) \\ &= (g')^{-1}\left(\frac{e^{(j-\alpha)t^*}\Omega(t^*)g'(m(t^*))}{\Omega(t)e^{(j-\alpha)t}}\right). \end{aligned} \tag{3.37}$$

$$\begin{aligned} S_0 &= \int_0^{t^*} e^{-\int_0^t j\,dx}\left[(g')^{-1}\left(\frac{e^{\int_t^{t^*} j\,dx}\Omega(t^*)e^{-\alpha t^*}g'(m(t^*))}{\Omega(t)e^{-\alpha t}}\right) - m(t)\right] dt \\ &= \int_0^{t^*} e^{-jt}\left[(g')^{-1}\left(\frac{e^{(j-\alpha)t^*}\Omega(t^*)g'(m(t^*))}{\Omega(t)e^{(j-\alpha)t}}\right) - m(t)\right] dt. \end{aligned} \tag{3.38}$$

$$\begin{aligned} S^*(t) &= e^{\int_0^t j\,dx}\left[S_0 + \int_0^t e^{-\int_0^z j\,dx}(m(z) - c^*(z))\right] dz \\ &= e^{jt}\left[S_0 + \int_0^t e^{-jz}(m(z) - c^*(z))dz\right]. \end{aligned} \tag{3.39}$$

We also define the following functions that will allow us to simplify the results of our calculations throughout the entire analysis:

$$\sigma(z, t) = \frac{e^{(j-\alpha)z}\Omega(z)g'(m(z))}{\Omega(t)e^{(j-\alpha)t}} \tag{3.40}$$

$$\Delta(t) = j - \alpha - \pi_t(t) + \frac{g''(m(t))m'(t)}{g'(m(t))} \tag{3.41}$$

$$\Psi(t) = \int_0^t e^{-jz}\left[\frac{g'(c^*(z))}{g''(c^*(z))}\right] dz. \tag{3.42}$$

We will use the following theorem to perform a sensitivity analysis on the model:

Theorem 3.2.1. *(Leibniz). Suppose* $f(x,t)$ *is a function where* $\frac{\partial f}{\partial t}$ *exists. Then*

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}dt + f(b(t),t)\cdot\frac{\partial b(t)}{t} - f(a(t),t)\cdot\frac{\partial a(t)}{t}.$$

3.2.1 Sensitivity of Model to Initial Value of Wealth

Sensitivity of Depletion of Wealth Time t*

First we work to determine how the time until the depletion of wealth, t^*, is affected by a change in the initial wealth, S_0. To do this, we must take the partial derivative of the function S_0 (equation 3.38) with respect to S_0. Recall that t^* is dependent upon the parameter S_0. We obtain

$$1 = e^{-jt^*}\left[(g')^{-1}\left(\frac{e^{(j-\alpha)t^*}\Omega(t^*)g'(m(t))}{\Omega(t^*)e^{(j-\alpha)t^*}}\right) - m(t^*)\right]\frac{\partial t^*}{\partial S_0} + \int_0^{t^*} e^{-jt}\frac{\partial (g')^{-1}(\sigma(t^*,t))}{\partial S_0}dt. \tag{3.43}$$

Notice that $(g')^{-1}\left(\frac{e^{(j-\alpha)t^*}\Omega(t^*)g'(m(t))}{\Omega(t^*)e^{(j-\alpha)t^*}}\right)$ is exactly $c^*(t^*)$ and that $c^*(t^*) = m(t^*)$. Thus we now have:

$$\begin{aligned}1 &= \int_0^{t^*} e^{-jt}\frac{\partial (g')^{-1}(\sigma(t^*,t))}{\partial S_0}dt\\ &= \int_0^{t^*} \frac{e^{-jt}}{g''((g')^{-1}(\sigma(t^*,t)))}\left[\frac{(j-\alpha)e^{(j-\alpha)t^*}\Omega(t^*)g'(m(t^*))}{\Omega(t)e^{(j-\alpha)t}} + \right.\\ &\qquad \left. + \frac{e^{(j-\alpha)t^*}\left(\Omega'(t^*)g'(m(t^*)) + \Omega(t^*)g''(m(t^*))m'(t^*)\right)}{\Omega(t)e^{(j-\alpha)t}}\right]\frac{\partial t^*}{\partial S_0}dt.\end{aligned} \tag{3.44}$$

Recall that $\Omega(t) = e^{-\int_0^t \pi_x(x)dx}$. Then $\Omega'(t) = -\pi_t(t)e^{-\int_0^t \pi_x(x)dx} = -\pi_t(t)\Omega(t)$. Equation (3.44) may then be rewritten and simplified in the following manner:

$$
\begin{aligned}
1 =& \int_0^{t^*} \frac{e^{-jt}}{g''((g')^{-1}(\sigma(t^*,t)))} \left[\frac{(j-\alpha)e^{(j-\alpha)t^*}\Omega(t^*)g'(m(t^*))}{\Omega(t)e^{(j-\alpha)t}} + \right. \\
&\left. + \frac{e^{(j-\alpha)t^*}\left(-\pi_{t^*}(t^*)\Omega(t^*)g'(m(t^*)) + \Omega(t^*)g''(m(t^*))m'(t^*)\right)}{\Omega(t)e^{(j-\alpha)t}} \right] \frac{\partial t^*}{\partial S_0} dt \\
=& \int_0^{t^*} \frac{e^{-jt}\left[j - \alpha - \pi_{t^*}(t^*) + \frac{g''(m(t^*))m'(t^*)}{g'(m(t^*))}\right] e^{(j-\alpha)t^*}\Omega(t^*)g'(m(t^*))}{g''((g')^{-1}(\sigma(t^*,t)))\Omega(t)e^{(j-\alpha)t}} \cdot \frac{\partial t^*}{\partial S_0} dt \\
=& \int_0^{t^*} \frac{e^{-jt}\Delta(t^*)\sigma(t^*,t)}{g''((g')^{-1}(\sigma(t^*,t)))} \cdot \frac{\partial t^*}{\partial S_0} dt. \qquad (3.45)
\end{aligned}
$$

Since $c^*(t) = (g')^{-1}(\sigma(t^*,t))$, we rewrite and simplify equation (3.45) to obtain the following:

$$
\begin{aligned}
1 =& \Delta(t^*)\frac{\partial t^*}{\partial S_0} \int_0^{t^*} e^{-jt} \frac{g'(c^*(t))}{g''(c^*(t))} dt \\
=& \Delta(t^*)\Psi(t^*)\frac{\partial t^*}{\partial S_0}. \qquad (3.46)
\end{aligned}
$$

Therefore, the way in which the time until depletion of wealth changes with respect to the initial wealth can be represented by

$$
\frac{\partial t^*}{\partial S_0} = \frac{1}{\Delta(t^*)\Psi(t^*)}. \qquad (3.47)
$$

Sensitivity of Optimal Consumption Path $c^*(t)$

We will perform a similar derivation to find out how a change in the initial wealth affects the optimal consumption path. The derivation is shown below:

$$\begin{aligned}\frac{\partial c^*(t)}{\partial S_0} =& \frac{\partial (g')^{-1}(\sigma(t^*,t))}{\partial S_0} = \frac{\partial (g')^{-1}(\sigma(t^*,t))}{\partial t^*}\left[\frac{\partial t^*}{\partial S_0}\right] \\ =& \frac{\Delta(t^*)\sigma(t^*,t)}{g''((g')^{-1}(\sigma(t^*,t)))} \cdot \frac{1}{\Delta(t^*)\Psi(t^*)} \\ =& \frac{\sigma(t^*,t)}{g''((g')^{-1}(\sigma(t^*,t)))\Psi(t^*)} \\ =& \frac{g'(c^*(t))}{g''(c^*(t))\Psi(t^*)}. \end{aligned} \tag{3.48}$$

Sensitivity of Optimal Wealth Path $S^*(t)$

To find how a change in the initial wealth affects the overall optimal wealth function associated with the optimal consumption path, we will take the partial derivative of $S^*(t)$ with respect to S_0.

$$\begin{aligned}\frac{\partial S^*(t)}{\partial S_0} =& e^{jt} - e^{jt}\int_0^t e^{-jz} \cdot \frac{\partial c^*(z)}{\partial S_0} dz \\ =& e^{jt}\left[1 - \int_0^t e^{-jz}\frac{g'(c^*(z))}{g''(c^*(z))\Psi(t^*)}\right] \\ =& e^{jt}\left[1 - \frac{\Psi(t)}{\Psi(t^*)}\right]. \end{aligned} \tag{3.49}$$

Interpretation of Sensitivity Analysis

Now that we have derived $\frac{\partial t^*}{\partial S_0}$, $\frac{\partial c^*(t)}{\partial S_0}$, and $\frac{\partial S^*(t)}{\partial S_0}$, we can see exactly how the time until depletion of wealth, the optimal consumption path, and the optimal accumulation of wealth path are affected by a change in the initial wealth value:

$$\frac{\partial t^*}{\partial S_0} = \frac{1}{\Delta(t^*)\Psi(t^*)} \tag{3.50}$$

$$\frac{\partial c^*(t)}{\partial S_0} = \frac{g'(c^*(t))}{g''(c^*(t))\Psi(t^*)} \tag{3.51}$$

$$\frac{\partial S^*(t)}{\partial S_0} = e^{jt}\left[1 - \frac{\Psi(t)}{\Psi(t^*)}\right]. \tag{3.52}$$

Since the utility function, g, is defined to be a concave function where $g'(c) > 0$ and $g''(c) < 0$, we know that $\Psi(t^*) < 0$. From this, we can determine that $\frac{\partial t^*}{\partial S_0} > 0$ for $\Delta(t^*) < 0$. We can also easily see that $\frac{\partial c^*(t)}{\partial S_0} > 0$ for all $t \in [0, t^*)$. Since $\Psi(t^*) < \Psi(t) < 0$, we can make the observation that $0 < \frac{\Psi(t)}{\Psi(t^*)} < 1$. Therefore, $\frac{\partial S^*(t)}{\partial S_0} > 0$ for all $t \in [0, t^*)$. This tells us that if a consumer's initial wealth is increased, the optimal consumption path, the accumulated wealth, and the time until depletion of wealth will all increase as well.

3.2.2 Sensitivity of Model to Discount Factor α

Sensitivity of Depletion of Wealth Time t*

The next step in the analysis is to study how changing the discount factor, α, affects the terminal wealth depletion time, optimal consumption path, and the associated optimal accumulation of wealth function. To do this, we will begin by taking the partial derivatives of S_0, $c^*(t)$, and $S^*(t)$ with respect to α. The derivation of $\frac{\partial S_0}{\partial \alpha}$ to find $\frac{\partial t^*}{\partial \alpha}$ is below:

$$
\begin{aligned}
0 =& e^{-jt^*}\left[(g')^{-1}\left(\frac{e^{(j-\alpha)t^*}\Omega(t^*)g'(m(t^*))}{\Omega(t^*)e^{(j-\alpha)t^*}}\right) - m(t^*)\right]\frac{\partial t^*}{\partial \alpha} + \int_0^{t^*} e^{-jt}\frac{\partial (g')^{-1}(\sigma(t^*,t))}{\partial \alpha}dt \\
=& e^{-jt}\left[c^*(t^*) - m(t^*)\right]\frac{\partial t^*}{\partial \alpha} + \int_0^{t^*} e^{-jt}\frac{\partial (g')^{-1}(\sigma(t^*,t))}{\partial \alpha}dt \\
=& \int_0^{t^*}\frac{e^{-jt}}{g''((g')^{-1}(\sigma(t^*,t)))}\Bigg(\frac{\Omega(t)e^{(j-\alpha)t}\left[e^{(j-\alpha)t^*}\left((j-\alpha)\frac{\partial t^*}{\partial \alpha} - t^*\right)\Omega(t^*)g'(m(t^*))\right.}{(\Omega(t)e^{(j-\alpha)t})^2} + \\
& + \frac{\left.e^{(j-\alpha)t^*}\left(\Omega(t^*)(-\pi_{t^*}(t^*))g'(m(t^*)) + \Omega(t^*)g''(m(t^*))m'(t^*)\right)\frac{\partial t^*}{\partial \alpha}\right]}{\left(\Omega(t)e^{(j-\alpha)t}\right)^2} + \\
& + \frac{te^{(j-\alpha)t^*}\Omega(t^*)g'(m(t^*))\Omega(t)e^{(j-\alpha)t}}{\left(\Omega(t)e^{(j-\alpha)t}\right)^2}\Bigg)dt \\
=& \int_0^{t^*}\frac{e^{-jt}\sigma(t^*,t)}{g''((g')^{-1}(\sigma(t^*,t)))}\left[\left(j-\alpha-\pi_{t^*}(t^*)+\frac{g''(m(t^*))m'(t^*)}{g'(m(t^*))}\right)\frac{\partial t^*}{\partial \alpha} - (t^*-t)\right]dt \\
=& \int_0^{t^*} e^{-jt}\left[\frac{\Delta(t^*)g'(c^*(t))}{g''(c^*(t))}\cdot\frac{\partial t^*}{\partial \alpha} - \frac{(t-t^*)g'(c^*(t))}{g''(c^*(t))}\right]dt \\
=& \Delta(t^*)\Psi(t^*)\frac{\partial t^*}{\partial \alpha} - \int_0^{t^*} e^{-jt}(t^*-t)\frac{g'(c^*(t))}{g''(c^*(t))}dt \qquad (3.53)
\end{aligned}
$$

Solving equation (3.53) for $\frac{\partial t^*}{\partial \alpha}$ gives us

$$
\frac{\partial t^*}{\partial \alpha} = \frac{\int_0^{t^*} e^{-jt}(t^*-t)\frac{g'(c^*(t))}{g''(c^*(t))}dt}{\Delta(t^*)\Psi(t^*)}. \qquad (3.54)
$$

Sensitivity of Optimal Consumption Path c*(t)

The next step will be to derive $\frac{\partial c^*(t)}{\partial \alpha}$ for $t \in [0, t^*)$ from equation (3.37):

$$\begin{aligned}
\frac{\partial c^*(t)}{\partial \alpha} =& \frac{\Delta(t^*)g'(c^*(t))}{g''(c^*(t))} \cdot \frac{\partial t^*}{\partial \alpha} - \frac{(t^*-t)g'(c^*(t))}{g''(c^*(t))} \\
=& \frac{g'(c^*(t))}{g''(c^*(t))\Psi(t^*)} \int_0^{t^*} e^{-jz}(t^*-z)\frac{g'(c^*(z))}{g''(c^*(z))}dz - (t^*-t)\frac{g'(c^*(t))}{g''(c^*(t))} \\
=& \frac{g'(c^*(t))}{g''(c^*(t))\Psi(t^*)} \left[\int_0^{t^*} e^{-jz}(t^*-z)\frac{g'(c^*(z))}{g''(c^*(z))}dz - \Psi(t^*)(t^*-t)\right] \\
=& \frac{g'(c^*(t))}{g''(c^*(t))\Psi(t^*)} \left[\int_0^{t^*} e^{-jz}(t^*-z)\frac{g'(c^*(z))}{g''(c^*(z))}dz - \int_0^{t^*} e^{-jz}\frac{g'(c^*(z))}{g''(c^*(z))}dz\,(t^*-t)\right] \\
=& \frac{g'(c^*(t))}{g''(c^*(t))\Psi(t^*)} \int_0^{t^*} e^{-jz}(t-z)\frac{g'(c^*(z))}{g''(c^*(z))}dz \\
=& \frac{g'(c^*(t))}{g''(c^*(t))\Psi(t^*)} \left[t\int_0^{t^*} e^{-jz}\frac{g'(c^*(z))}{g''(c^*(z))}dz - \int_0^{t^*} e^{-jz}z\frac{g'(c^*(z))}{g''(c^*(z))}dz\right] \\
=& \frac{g'(c^*(t))}{g''(c^*(t))}\left[t\Psi(t^*) - \int_0^{t^*} e^{-jz}z\frac{g'(c^*(z))}{g''(c^*(z))}dz\right].
\end{aligned} \tag{3.55}$$

Define $\xi = \frac{1}{\Psi(t^*)}\int_0^{t^*} e^{-jz}z\frac{g'(c^*(z))}{g''(c^*(z))}dz$. Then equation (3.55) becomes

$$\frac{\partial c^*(t)}{\partial \alpha} = \frac{g'(c^*(t))}{g''(c^*(t))}[t-\xi]. \tag{3.56}$$

Sensitivity of Optimal Wealth Path $S^*(t)$

The final derivation of $\frac{\partial S^*(t)}{\partial \alpha}$ for $t \in [0, t^*)$ is shown below:

$$\begin{aligned}
\frac{\partial S^*(t)}{\partial \alpha} =& -e^{jt}\int_0^t e^{-jx}\frac{\partial c^*(x)}{\partial \alpha}dx \\
=& -e^{jt}\int_0^t \left(e^{-jx}\frac{g'(c^*(x))}{g''(c^*(x))\Psi(t^*)}\cdot\int_0^{t^*} e^{-jz}(x-z)\frac{g'(c^*(z))}{g''(c^*(z))}dz\right)dx \\
=& \frac{e^{jt}}{\Psi(t^*)}\int_0^t \left(e^{-jx}\frac{g'(c^*(x))}{g''(c^*(x))}\cdot\int_0^{t^*} e^{-jz}(z-x)\frac{g'(c^*(z))}{g''(c^*(z))}dz\right)dx \\
=& \frac{e^{-jt}}{\Psi(t^*)}\int_0^t e^{-jx}\frac{g'(c^*(x))}{g''(c^*(x))}\left[\int_0^{t^*} e^{-jz}z\frac{g'(c^*(z))}{g''(c^*(z))}dz - x\Psi(t^*)\right]dx \\
=& \frac{e^{-jt}}{\Psi(t^*)}\left[\int_0^t e^{-jx}\frac{g'(c^*(x))}{g''(c^*(x))}dx\int_0^{t^*} e^{-jz}z\frac{g'(c^*(z))}{g''(c^*(z))}dz - \Psi(t^*)\int_0^t e^{-jx}x\frac{g'(c^*(x))}{g''(c^*(x))}dx\right] \\
=& e^{-jt}\left[\frac{\Psi(t)}{\Psi(t^*)}\int_0^{t^*} e^{-jz}z\frac{g'(c^*(z))}{g''(c^*(z))}dz - \int_0^t e^{-jx}x\frac{g'(c^*(x))}{g''(c^*(x))}dx\right]. \qquad (3.57)
\end{aligned}$$

Interpretation of Sensitivity Analysis

We can now see from $\frac{\partial t^*}{\partial \alpha}$, $\frac{\partial c^*(t)}{\partial \alpha}$, and $\frac{\partial S^*(t)}{\partial \alpha}$ exactly how the time until depletion of wealth, the optimal consumption path, and the associated optimal accumulation of wealth function are affected by a change in the discount factor:

$$\frac{\partial t^*}{\partial \alpha} = \frac{\int_0^{t^*} e^{-jt}(t^*-t)\frac{g'(c^*(t))}{g''(c^*(t))}dt}{\Delta(t^*)\Psi(t^*)} \qquad (3.58)$$

$$\frac{\partial c^*(t)}{\partial \alpha} = \frac{g'(c^*(t))}{g''(c^*(t))}[t-\xi] \qquad (3.59)$$

$$\frac{\partial S^*(t)}{\partial \alpha} = e^{-jt}\left[\frac{\Psi(t)}{\Psi(t^*)}\int_0^{t^*} e^{-jz}z\frac{g'(c^*(z))}{g''(c^*(z))}dz - \int_0^t e^{-jx}x\frac{g'(c^*(x))}{g''(c^*(x))}dx\right]. \qquad (3.60)$$

Just as before, the utility function, g, is defined to be a concave function where $g'(c) > 0$ and $g''(c) < 0$. This will again force $\Psi(t^*) < 0$. Notice then that $\frac{\partial t^*}{\partial \alpha}$ must be negative when $\Delta(t^*)$ is negative. Since $\frac{\partial t^*}{\partial \alpha} < 0$, we know that if we increase the discount factor α, the time until depletion of wealth will decrease. From equation (3.59), we can easily see that if $t \geqslant \xi$, then $\frac{\partial c^*(t)}{\partial \alpha} \leqslant 0$, and if $t \leqslant \xi$, then $\frac{\partial c^*(t)}{\partial \alpha} \geqslant 0$ for $t \in [0, t^*)$. This tells us that an increase in the discount factor α will cause an individual's optimal consumption path to be greater before time ξ and lower after time ξ. Now notice that for $t \in (0, t^*)$, $\int_0^{t^*} e^{-jz} z \frac{g'(c^*(z))}{g''(c^*(z))} dz < \int_0^t e^{-jz} z \frac{g'(c^*(z))}{g''(c^*(z))} dz < 0$ and that $\frac{\Psi(t)}{\Psi(t^*)} > 0$. Then it is clear that $\frac{\partial S^*(t)}{\partial \alpha} < 0$ for $t \in (0, t^*)$. Therefore, if there is an increase in α, the optimal accumulation of wealth function will be lower [4].

3.3 Model Implementation

In this section, we will study a direct application of the model for a recently retired individual. Assume that the individual is currently 65 years old and has a constant income stream $m(t) = m$ from a retirement annuity. We will also assume a discount function of $\alpha(t) = e^{-\alpha t}$, an interest rate of $j(t) = j$, and a maximum possible lifetime of $\bar{T}$. Let us also define the utility function g with the Constant Relative Risk Aversion (CRRA) utility function:

$$g(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \gamma \neq 1 \\ \ln(c) & \gamma = 1 \end{cases}, \tag{3.61}$$

where c represents the consumption function and $\gamma \in \mathbb{R}$ represents the relative risk aversion coefficient [3]. By using the CRRA utility function, we can account for approximately how risk averse the individual is and how that may impact their optimal consumption and wealth paths. The higher the value of γ, the more risk averse the individual.

3.3.1 Sensitivity Analysis on the Relative Risk Aversion Coefficient

Before progressing deeper into the analysis, we must first rederive $c^*(t)$, $S^*(t)$, S_0, $\Delta(t)$, and $\Psi(t)$ with our CRRA utility function g. Begin by noting that $g'(c) = c^{-\gamma}$, $(g')^{-1}(c) =$

$(1/c)^{1/\gamma}$, and $g''(c) = -\gamma c^{-\gamma-1}$ for all $\gamma \in \mathbb{R}$. The modified functions are shown below:

$$
\begin{aligned}
c^*(t) =& (g)^{-1}\left(\frac{e^{(j-\alpha)t^*}\Omega(t^*)g'(m)}{\Omega(t)e^{(j-\alpha)t}}\right) \\
=& \left(\frac{\Omega(t)e^{(j-\alpha)t}}{e^{(j-\alpha)t^*}\Omega(t^*)(m)^{-\gamma}}\right)^{1/\gamma} \\
=& \left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)^{1/\gamma}\cdot m
\end{aligned}
\tag{3.62}
$$

$$
\begin{aligned}
S^*(t) =& e^{jt}\left[S_0 + \int_0^t e^{-jz}\left(m - c^*(z)\right)dz\right] \\
=& e^{jt}\left[S_0 + \int_0^t e^{-jz}\left(m - \left(\frac{\Omega(z)}{\Omega(t^*)}e^{(j-\alpha)(z-t^*)}\right)\right)dz\right]
\end{aligned}
\tag{3.63}
$$

$$
\begin{aligned}
S_0 =& \int_0^{t^*} e^{-jt}\left[(g')^{-1}\left(\frac{e^{(j-\alpha)t^*}\Omega(t^*)g'(m)}{\Omega(t)e^{(j-\alpha)t}}\right) - m\right]dt \\
=& \int_0^{t^*} e^{-jt}\left[\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)^{1/\gamma}\cdot m - m\right]dt
\end{aligned}
\tag{3.64}
$$

$$
\begin{aligned}
\Delta(t) =& -\pi_t(t) - \alpha + j - \frac{m'(t)g''(m)}{g'(m)} \\
=& -\pi_t(t) - \alpha + j
\end{aligned}
\tag{3.65}
$$

$$
\begin{aligned}
\Psi(t) =& \int_0^t e^{-jz}\left[\frac{g'(c^*(z))}{g''(c^*(z))}\right]dz \\
=& \int_0^t e^{-jz}\left[\frac{c^*(z)^{-\gamma}}{-\gamma c^*(z)^{-\gamma-1}}\right]dz \\
=& \frac{-1}{\gamma}\int_0^t e^{-jz}c^*(z)dz.
\end{aligned}
\tag{3.66}
$$

We may now begin our analysis by deriving the partial derivatives of t^*, $c^*(t)$, and $S^*(t)$ with respect to the relative risk aversion coefficient γ. This will help us determine how sensitive the terminal depletion of wealth time, optimal consumption path, and optimal accumulation of wealth path are all affected by a small change in γ.

Sensitivity of Depletion of Wealth Time t*

The derivation of $\frac{\partial t^*}{\partial \gamma}$ will involve taking the partial derivative of equation (3.64) with respect to γ:

$$\begin{aligned} 0 =& e^{-jt^*}\left[\left(\frac{\Omega(t^*)}{\Omega(t^*)}e^{(j-\alpha)(t^*-t^*)}\right)^{1/\gamma} m - m\right]\left(\frac{\partial t^*}{\partial \gamma}\right) + \\ &+ \int_0^{t^*} e^{-jt}\left[\frac{\partial}{\partial \gamma}\left(\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)^{1/\gamma}\right) m - m\right] dt \\ =& \int_0^{t^*} e^{-jt}\left[\frac{\partial}{\partial \gamma}\left(\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)^{1/\gamma}\right) m - m\right] dt. \end{aligned} \tag{3.67}$$

Before moving any further, we must use logarithmic differentiation to find the partial derivative of $\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)^{1/\gamma}$ with respect to γ:

$$\begin{aligned} y =& \left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)^{1/\gamma} \\ \ln y =& \frac{1}{\gamma}\ln\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right) \\ \frac{\partial}{\partial \gamma}(\ln y) =& \frac{\partial}{\partial \gamma}\left(\frac{1}{\gamma}\ln\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)\right) \\ \frac{1}{y}\cdot\frac{\partial y}{\partial \gamma} =& \frac{-1}{\gamma^2}\ln\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right) + \\ &+ \frac{1}{\gamma}\cdot\frac{1}{\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}}\left[\frac{\Omega(t)(\pi_{t^*}(t^*))\Omega(t^*)}{(\Omega(t^*))^2}e^{(j-\alpha)(t-t^*)} + \right. \\ &\left. + (\alpha - j)\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right]\left(\frac{\partial t^*}{\partial \gamma}\right) \\ \frac{1}{y}\cdot\frac{\partial y}{\partial \gamma} =& \frac{-1}{\gamma^2}\ln\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right) + \frac{1}{\gamma}(\pi_{t^*}(t^*) + \alpha - j)\left(\frac{\partial t^*}{\partial \gamma}\right) \end{aligned} \tag{3.68}$$

Solving equation (3.68) for $\frac{\partial y}{\partial \gamma}$ gives us the derivative of $\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)^{1/\gamma}$ with respect to γ:

$$\frac{\partial y}{\partial \gamma} = \left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)^{1/\gamma}\left[\frac{-1}{\gamma^2}\ln\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right) - \frac{\Delta(t^*)}{\gamma}\left(\frac{\partial t^*}{\partial \gamma}\right)\right]. \tag{3.69}$$

Now that we have derived the partial derivative of $\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)^{1/\gamma}$ with respect to γ (equation (3.69)), we may now complete the derivation of $\frac{\partial t^*}{\partial \gamma}$. Continuing from equation (3.67), we have

$$\begin{aligned}
0 =& \int_0^{t^*} e^{-jt}\left[\frac{\partial}{\partial \gamma}\left(\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)^{1/\gamma} m - m\right)\right]dt \\
=& \int_0^{t^*} e^{-jt} m\left[\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)^{1/\gamma}\left[\frac{-1}{\gamma^2}\ln\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right) - \frac{\Delta(t^*)}{\gamma}\left(\frac{\partial t^*}{\partial \gamma}\right)\right]\right]dt \\
=& \frac{-1}{\gamma^2}\int_0^{t^*} e^{-jt}\cdot c^*(t)\ln\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)dt - \frac{\Delta(t^*)}{\gamma}\left(\frac{\partial t^*}{\partial \gamma}\right)\int_0^{t^*} e^{-jt}c^*(t)dt \\
=& \frac{-1}{\gamma^2}\int_0^{t^*} e^{-jt}\cdot c^*(t)\ln\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)dt + \Delta(t^*)\Psi(t^*)\left(\frac{\partial t^*}{\partial \gamma}\right).
\end{aligned} \tag{3.70}$$

Thus,

$$\frac{\partial t^*}{\partial \gamma} = \frac{\int_0^{t^*} e^{-jt}c^*(t)\ln\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)dt}{\gamma^2\Delta(t^*)\Psi(t^*)}. \tag{3.71}$$

Sensitivity of Optimal Consumption Path c*(t)

By using equation (3.69), we may also derive $\frac{\partial c^*(t)}{\partial \gamma}$:

$$\begin{aligned}
\frac{\partial c^*(t)}{\partial \gamma} =& m \left(\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right)^{1/\gamma} \left[\frac{-1}{\gamma^2} \ln \left(\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right) - \frac{\Delta(t^*)}{\gamma} \left(\frac{\partial t^*}{\partial \gamma} \right) \right] \\
=& \frac{-c^*(t)}{\gamma^2} \ln \left(\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right) - \Delta(t^*) \left(\frac{c^*(t)}{\gamma} \right) \left(\frac{\partial t^*}{\partial \gamma} \right) \\
=& \frac{-c^*(t)}{\gamma^2} \ln \left(\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right) + \\
& - \Delta(t^*) \left(\frac{c^*(t)}{\gamma} \right) \left(\frac{\int_0^{t^*} e^{-jz} c^*(z) \ln \left(\frac{\Omega(z)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right) dt}{\gamma^2 \Delta(t^*) \Psi(t^*)} \right) \\
=& \frac{-c^*(t)}{\gamma^2} \ln \left(\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right) + \frac{c^*(t) \int_0^{t^*} e^{-jz} c^*(z) \ln \left(\frac{\Omega(z)}{\Omega(t^*)} e^{(j-\alpha)(z-t^*)} \right) dt}{\gamma^2 \int_0^{t^*} e^{-jz} c^*(z) dz} \\
=& c^*(t) \left[\frac{- \int_0^{t^*} e^{-jz} c^*(z) dz \cdot \ln \left(\frac{\Omega(t)}{\Omega(t^*)} e^{(j-\alpha)(t-t^*)} \right)}{\gamma \int_0^{t^*} e^{-jz} c^*(z) dz} + \right. \\
& \left. + \frac{\int_0^{t^*} e^{-jz} c^*(z) \ln \left(\frac{\Omega(z)}{\Omega(t^*)} e^{(j-\alpha)(z-t^*)} \right) dz}{\gamma^2 \int_0^{t^*} e^{-jz} c^*(z) dz} \right] \\
=& c^*(t) \left[\frac{- \int_0^{t^*} e^{-jz} c^*(z) dz \cdot (\ln \Omega(t) - \ln \Omega(t^*) + jt - jt^* - \alpha t + \alpha t^*)}{\gamma^2 \int_0^{t^*} e^{-jz} c^*(z) dz} + \right. \\
& \left. + \frac{\int_0^{t^*} e^{-jz} c^*(z) \left(\ln \Omega(z) - \ln \Omega(t^*) + jz - jt^* - \alpha z + \alpha t^* \right) dz}{\gamma^2 \int_0^{t*} e^{-jz} c^*(z) dz} \right].
\end{aligned} \tag{3.72}$$

Through just a bit of simplification, we can rewrite equation (3.72) as the following:

$$\frac{\partial c^*(t)}{\partial \gamma} = c^*(t) \left[\frac{\int_0^{t^*} e^{-jz} c^*(z) \ln \left(\frac{\Omega(z)}{\Omega(t)} e^{(j-\alpha)(z-t)} \right) dz}{\gamma^2 \int_0^{t^*} e^{-jz} c^*(z) dz} \right]. \tag{3.73}$$

Sensitivity of Optimal Wealth Path S*(t)

The derivation of $\frac{\partial S^*(t)}{\partial \gamma}$ is fairly straightforward and is shown below:

$$\begin{aligned}\frac{\partial S^*(t)}{\partial \gamma} &= -e^{jt}\int_0^t e^{-jx}\frac{\partial c^*(x)}{\partial \gamma}dx \\ &= -e^{jt}\int_0^t e^{-jx}\left(\frac{c^*(x)\int_0^{t^*} e^{-jz}c^*(z)\ln\left(\frac{\Omega(z)}{\Omega(x)}e^{(j-\alpha)(z-x)}\right)dz}{\gamma^2\int_0^{t^*} e^{-jz}c^*(z)dz}\right)dx \\ &= \frac{-e^{jt}\int_0^t e^{-jx}c^*(x)\int_0^{t^*} e^{-jz}c^*(z)\ln\left(\frac{\Omega(z)}{\Omega(x)}e^{(j-\alpha)(z-x)}\right)dz\,dx}{\gamma^2\int_0^{t^*} e^{-jz}c^*(z)dz}. \end{aligned} \tag{3.74}$$

Interpretation of Sensitivity Analysis

From our derivations of $\frac{\partial t^*}{\partial \gamma}$, $\frac{\partial c^*(t)}{\partial \gamma}$, and $\frac{\partial S^*(t)}{\partial \gamma}$, we can see how the time until depletion of wealth, optimal consumption path, and optimal accumulation path are affected by a change in the relative risk aversion coefficient γ.

$$\frac{\partial t^*}{\partial \gamma} = \frac{\int_0^{t^*} e^{-jt}c^*(t)\ln\left(\frac{\Omega(t)}{\Omega(t^*)}e^{(j-\alpha)(t-t^*)}\right)dt}{\gamma^2\Delta(t^*)\Psi(t^*)} \tag{3.75}$$

$$\frac{\partial c^*(t)}{\partial \gamma} = c^*(t)\left[\frac{\int_0^{t^*} e^{-jz}c^*(z)\ln\left(\frac{\Omega(z)}{\Omega(t)}e^{(j-\alpha)(z-t)}\right)dz}{\gamma^2\int_0^{t^*} e^{-jz}c^*(z)dz}\right] \tag{3.76}$$

$$\frac{\partial S^*(t)}{\partial \gamma} = \frac{-e^{jt}\int_0^t e^{-jx}c^*(x)\int_0^{t^*} e^{-jz}c^*(z)\ln\left(\frac{\Omega(z)}{\Omega(x)}e^{(j-\alpha)(z-x)}\right)dz\,dx}{\gamma^2\int_0^{t^*} e^{-jz}c^*(z)dz}. \tag{3.77}$$

Notice that $\Omega(t)e^{(j-\alpha)t} > \Omega(t^*)e^{(j-\alpha)t^*}$ for $t \in [0, t^*)$ and that $\pi_t'(t) > 0$ for $t \in [0, t^*)$. Given the assumptions of the model and recalling that $\Psi(t^*) \leqslant 0$, we can see from equation (3.75) that $\frac{\partial t^*}{\partial \gamma}$ must be positive when $\Delta(t^*)$ is negative. This tells us that if we are more risk averse our time until depletion of wealth will increase. Now let us define ω as the time at which $\int_0^{t^*} e^{-jz}c^*(z)\ln\left(\frac{\Omega(z)}{\Omega(\omega)}e^{(j-\alpha)(z-\omega)}\right)dz = 0$ for $c^* > 0$. Then $\frac{\partial c^*(t)}{\partial \gamma} \geqslant 0$ if $t \geqslant \omega$ and $\frac{\partial c^*(t)}{\partial \gamma} \leqslant 0$ if $t \leqslant \omega$. This indicates that individuals who have a higher relative risk aversion coefficient tend to consume less before time ω and consume more afterwards. The opposite is true for those who are less risk averse. We can also see that $\frac{\partial S^*(t)}{\partial \gamma} > 0$ for $t \in (0, t^*)$, meaning individuals that are more risk averse will retain their wealth longer than individuals that are less risk averse.

3.3.2 MATLAB Simulation

The Gompertz probability distribution is a commonly used distribution in actuarial mortality models to determine the probability of survival and death. For this reason, we will integrate the Gompertz probability density function,

$$f(x) = ae^{bx}e^{-\frac{a}{b}\left(e^{bx}-1\right)}, \tag{3.78}$$

so that we can properly define our survival function $\Omega(t)$[7]. Since $\Omega(t)$ is defined to be the probability of an individual surviving past a certain time, we will integrate the Gompertz probability density function in the following manner:

$$\begin{aligned} \Omega(t) =& 1 - \int_0^t ae^{bx}e^{-\frac{a}{b}\left(e^{bx}-1\right)}dx \\ =& 1 - \left(-e^{\frac{-a}{b}\left(e^{bx}-1\right)}\Big|_0^t\right) \\ =& e^{\frac{-a}{b}\left(e^{bt}-1\right)}. \end{aligned} \tag{3.79}$$

For the purpose of simulating our model in MATLAB, we will set $a = 0.000081$ and $b = 0.087$, just as was done in Leung's 1994 paper, "Uncertain Lifetime, the Theory of the Consumer, and the Life Cycle Hypothesis." In the simulated model, we will also be using a selection of the parameters from the same paper shown in Table 3.1 [3].

		t^*		
		$S(65)/m$		
γ	α	1	5	10
3	0.10	73	80	84
	0.05	75	83	88
	0.03	77	85	89
	0.01	79	87	91
1	0.10	70	74	76
	0.05	71	77	80
	0.03	73	79	82
	0.01	75	81	84
0.5	0.10	68	71	73
	0.05	70	73	76
	0.03	71	75	78
	0.01	73	78	80

Table 3.1: A selection of parameters from Leung's "Uncertain Lifetime, The Theory of the Consumer, and the Life Cycle Hypothesis"

For model simplicity, we will define $m = 1$. Displayed below are figures showing the simulated optimal consumption and associated accumulated wealth functions for each of the parameters in Table 3.1 and for $j = 0.03$.

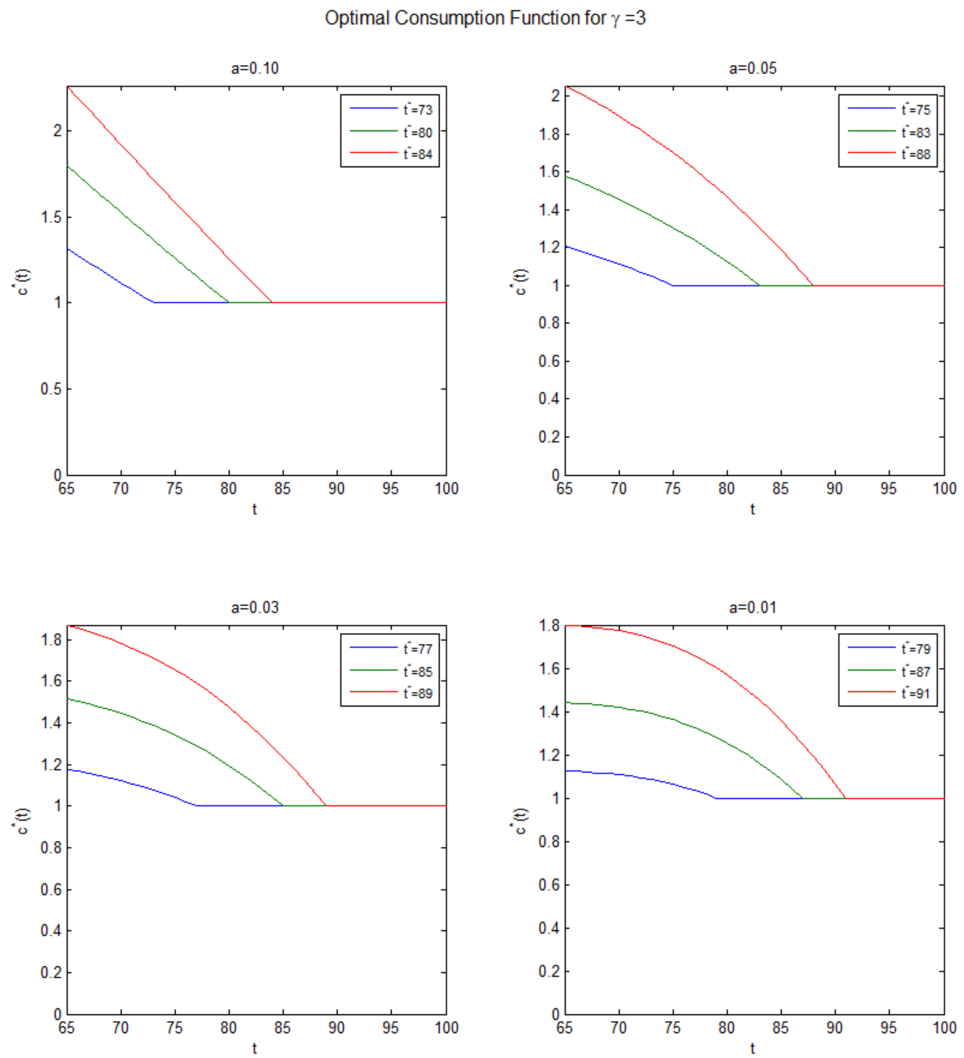

Figure 3.1: Each plot below displays the optimal consumption path for an individual with a specified t^* found from the equation for S_0, $\alpha = 0.10, 0.05, 0.03$, and 0.01, and $\gamma = 3$.

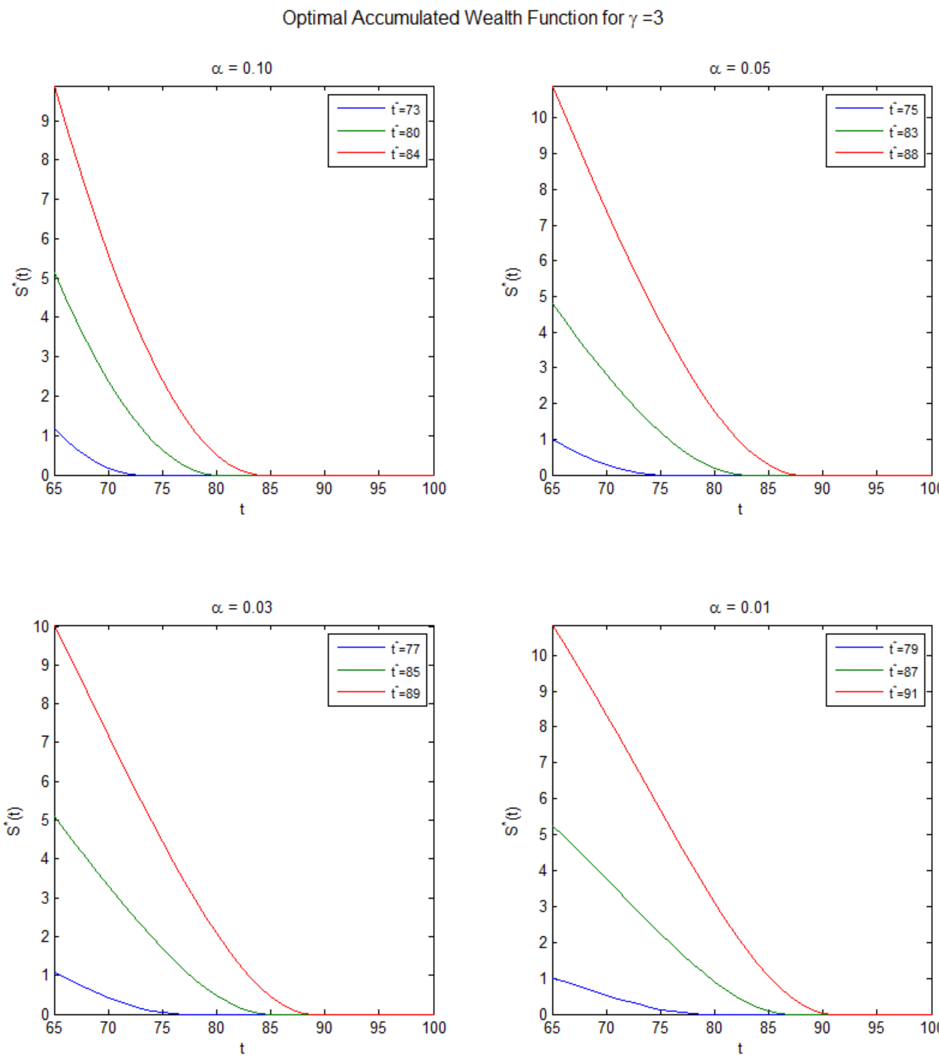

Figure 3.2: Each plot below displays the associated optimal accumulation of wealth path for an individual with a specified t^* found from the equation for S_0, $\alpha = 0.10, 0.05, 0.03$, and 0.01, and $\gamma = 3$.

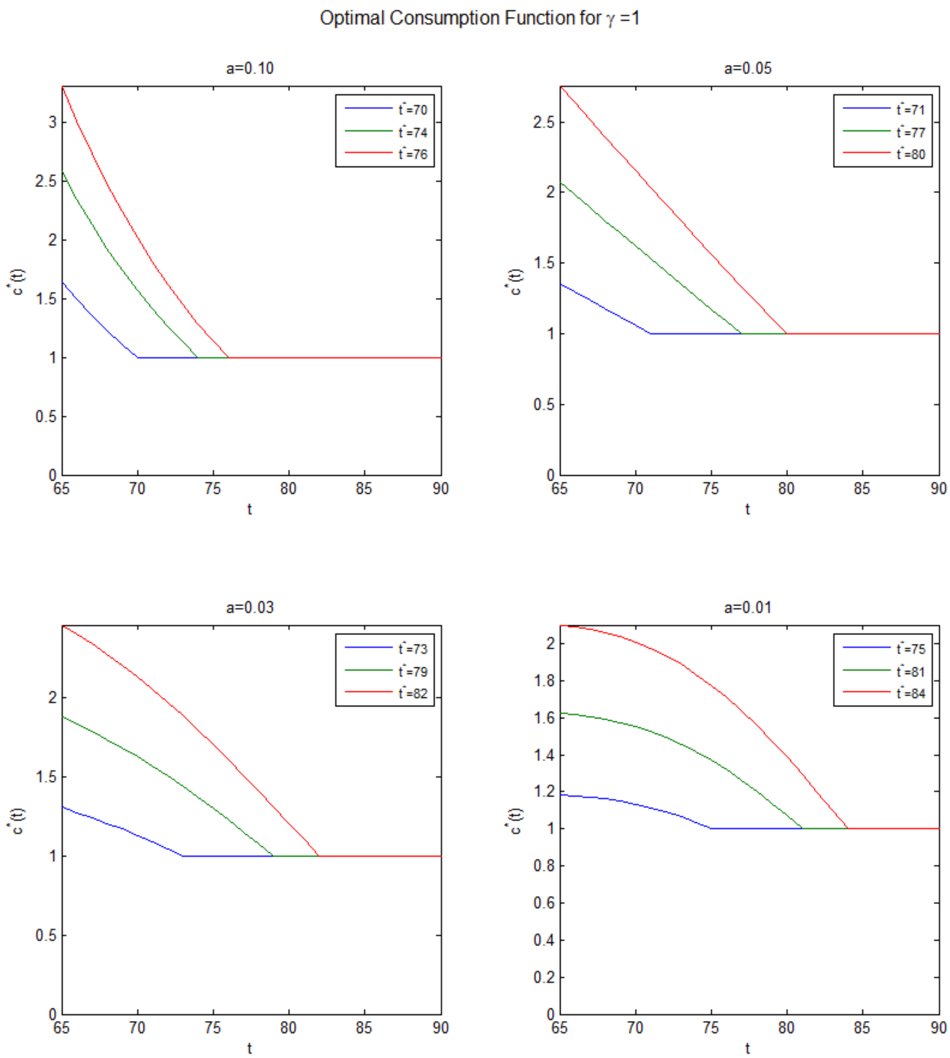

Figure 3.3: Each plot below displays the optimal consumption path for an individual with a specified t^* found from the equation for S_0, $\alpha = 0.10, 0.05, 0.03$, and 0.01, and $\gamma = 1$.

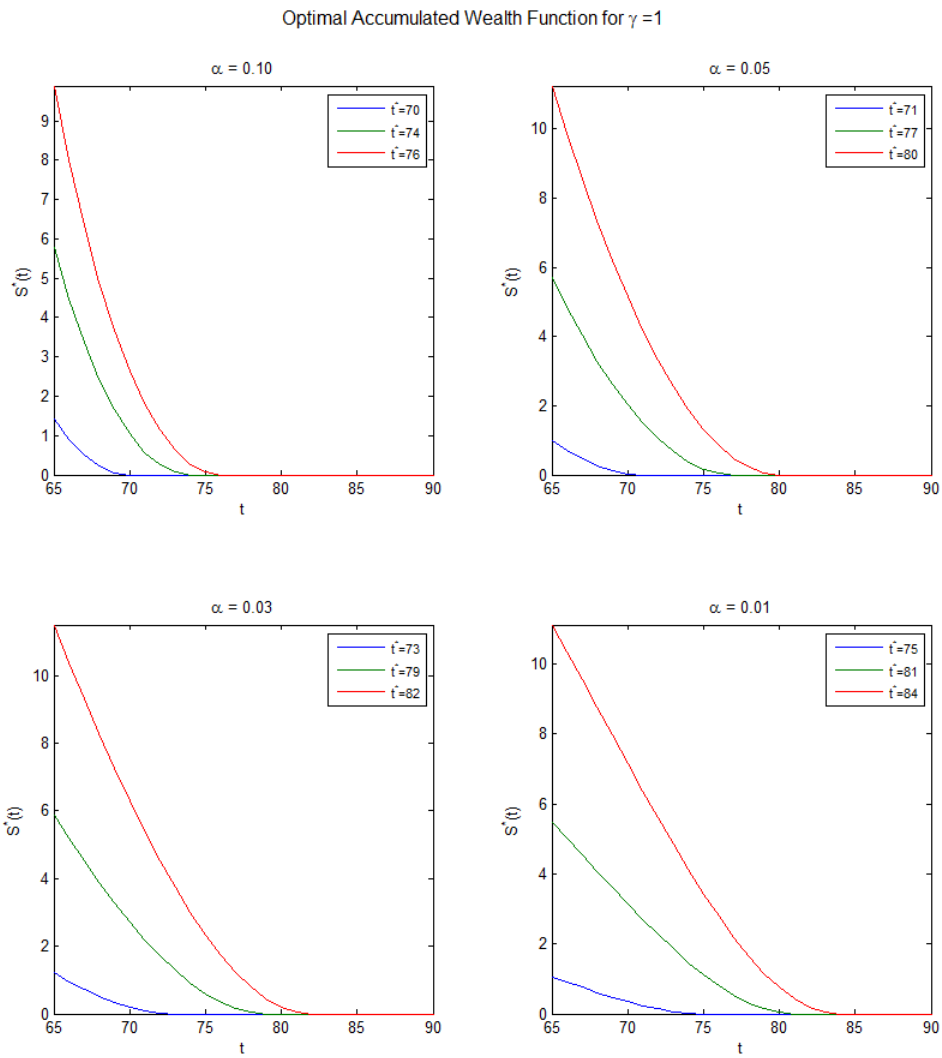

Figure 3.4: Each plot below displays the associated optimal accumulation of wealth path for an individual with a specified t^* found from the equation for S_0, $\alpha = 0.10, 0.05, 0.03$, and 0.01, and $\gamma = 1$.

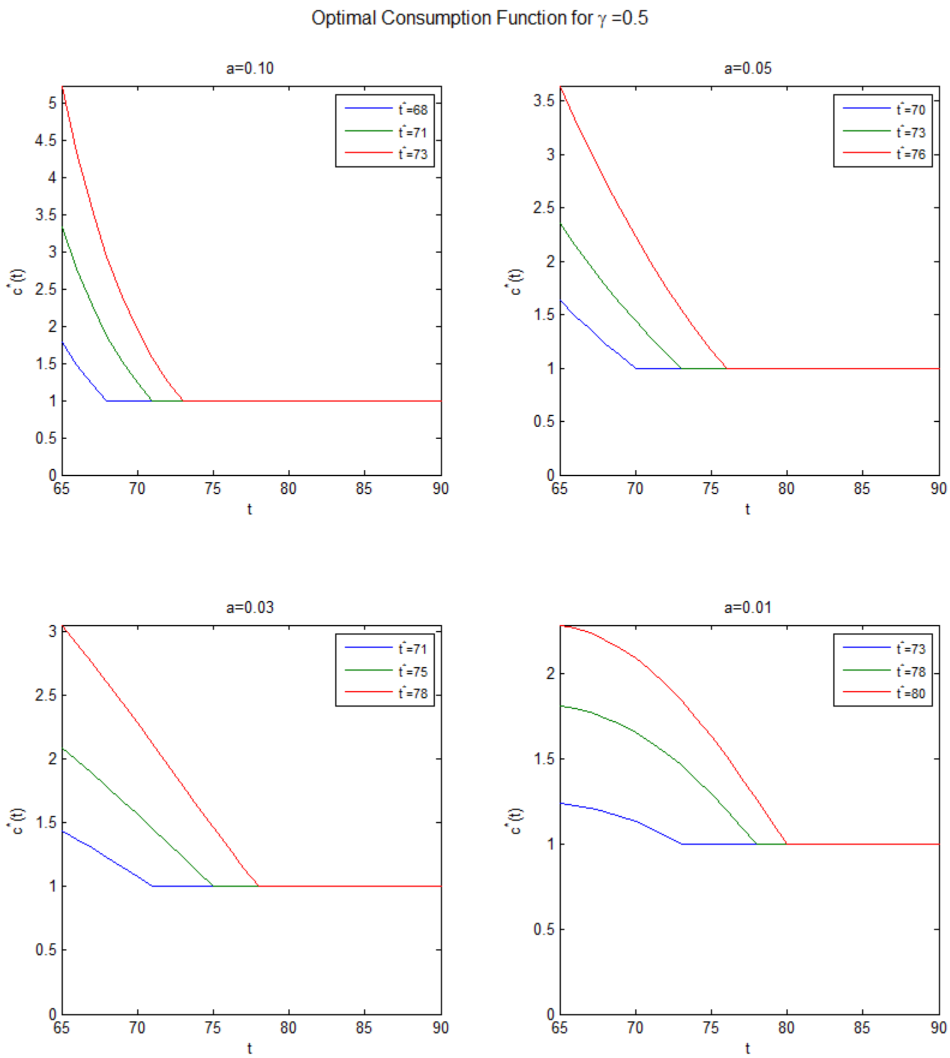

Figure 3.5: Each plot below displays the optimal consumption path for an individual with a specified t^* found from the equation for S_0, $\alpha = 0.10, 0.05, 0.03$, and 0.01, and $\gamma = 0.5$.

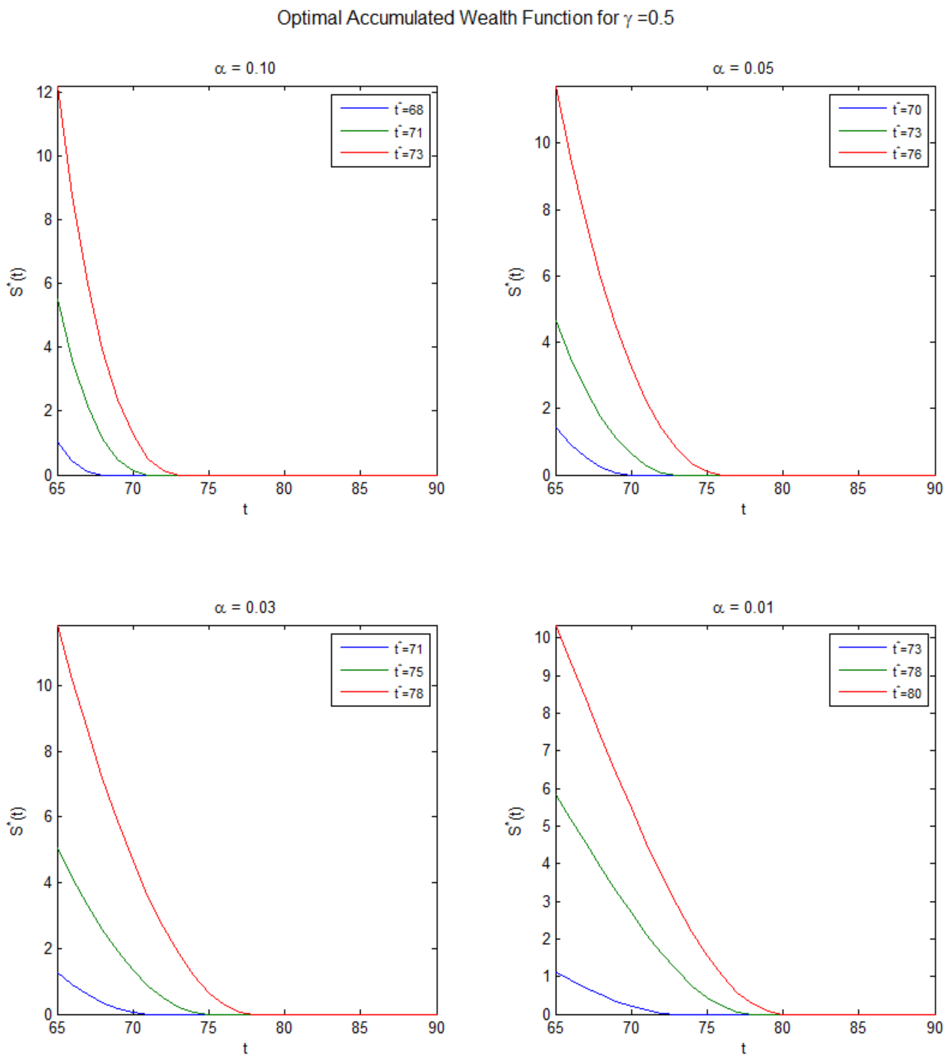

Figure 3.6: Each plot below displays the associated optimal accumulation of wealth path for an individual with a specified t^* found from the equation for S_0, $\alpha = 0.10, 0.05, 0.03$, and 0.01, and $\gamma = 0.5$.

Observe from figures 3.1, 3.3, and 3.5 that when γ is lower in value, the steeper the consumption function. This indicates that when a consumer is less risk averse and perhaps even risk seeking, the consumer will tend to spend more money right after retirement. On the other hand, if a consumer is very risk averse, they will tend to spend quite a bit less money right after retirement. The dynamics of the accumulation of wealth functions go hand-in-hand with the dynamics of the consumption functions. We can see from figures 3.2, 3.4, and 3.6 that when γ is lower in value, the accumulation of wealth function decreases at a much quicker rate than when γ is greater in value. With this, we can also see that the depletion of wealth time t^* is much lower for a small value of γ than with a large value of γ. In other words, an individual tends to ration their savings when they are more risk averse, causing their time until depletion of wealth to increase.

Chapter 4

Conclusion

Throughout the course of this thesis, we discussed the mathematics behind optimal control theory and Pontryagin's Maximum Principle, as well as applied optimal control theory to the life-cycle savings model under uncertain lifetime. From our sensitivity analysis, we are able to conclude that an individual with a higher initial wealth value will have a higher optimal consumption path and accumulated wealth path. We also found that the changes in dynamics of the consumption path when increasing the discount factor were not monotonic. In fact if the discount factor is increased, then the individual will have a higher optimal consumption path until a certain point in time. Once that time is reached, the individual's optimal consumption path will be lower. An increase in the discount factor also resulted in a decrease in the accumulation of wealth function. By simulating the model in MATLAB, we are able to not only visualize the dynamics of the model, but we are also able to verify some of the findings from the sensitivity analysis. From this paper, we can see exactly how we can control our consumption when given an initial wealth, income function, interest rate, and discount factor so that we can optimize the Fisher Utility function over an entire lifetime.

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Appendix 1

1 MATLAB Code For Optimal Consumption Function

```
j=0.03;
gamma=[3 1 0.5 0.1];
a=[0.10 0.05 0.03 0.01];
tstar=[73 80 84; 75 83 88; 77 85 89; 79 87 91; 70 74 76; 71 77 80; 73 79 82; 75 81 84; 68 71 73; 70 73 76; 71 75 78; 73 78 80; 66 67 68; 67 69 70; 68 70 71; 69 72 73];
m=1;

t=linspace(65,140,100);

figure
for i=1:1 %gamma
    for q=1:1 %a
        for k=1:3 %columns of tstar
            t=ones(76,1);
            y=ones(76,1);
            for p=1:76
                t(p)=p+64;
```

```
                if (t(p) < tstar(4.*(i-1)+q,k))
                    y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)+q,k))))^(1/gamma(i))*m
                else
                    y(p)=m
                end
            end
            subplot(2,2,1)
            plot(t,y)
            hold all
            legend({'t^*=73','t^*=80','t^*=84'},'FontSize',8)
            title('a=0.10')
            xlabel('t')
            ylabel('c^*(t)')
            axis([65 100 0 inf])
        end
    end
end

for i=1:1 %gamma
    for q=2:2 %a
        for k=1:3 %columns of tstar
            t=ones(76,1);
            y=ones(76,1);
            for p=1:76
```

```
                t(p)=p+64;
                if (t(p) < tstar(4.*(i-1)+q,k))
                    y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
                        exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
                        ))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
                        +q,k))))^(1/gamma(i))*m
                else
                    y(p)=m
                end
            end
            subplot(2,2,2)
            plot(t,y)
            hold all
            legend({'t^*=75','t^*=83','t^*=88'},'FontSize',8)
            xlabel('t')
            ylabel('c^*(t)')
            title('a=0.05')
            axis([65 100 0 inf])
        end
    end
end

for i=1:1 %gamma
    for q=3:3 %a
        for k=1:3 %columns of tstar
            t=ones(76,1);
            y=ones(76,1);
```

```
            for p=1:76
                t(p)=p+64;
                if (t(p) < tstar(4.*(i-1)+q,k))
                    y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)+q,k))))^(1/gamma(i))*m
                else
                    y(p)=m
                end
            end
            subplot(2,2,3)
            plot(t,y)
            hold all
            legend({'t^*=77','t^*=85','t^*=89'},'FontSize',8)
            xlabel('t')
            ylabel('c^*(t)')
            title('a=0.03')
            axis([65 100 0 inf])
        end
    end
end

for i=1:1 %gamma
    for q=4:4 %a
        for k=1:3 %columns of tstar
            t=ones(76,1);
```

```
                y=ones(76,1);
                for p=1:76
                    t(p)=p+64;
                    if (t(p) < tstar(4.*(i-1)+q,k))
                        y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
                            exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
                            ))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
                            +q,k))))^(1/gamma(i))*m
                    else
                        y(p)=m
                    end
                end
                subplot(2,2,4)
                plot(t,y)
                hold all
                legend({'t^*=79','t^*=87','t^*=91'},'FontSize',8)
                xlabel('t')
                ylabel('c^*(t)')
                title('a=0.01')
                axis([65 100 0 inf])
            end
        end
end
annotation('textbox', [0 0.9 1 0.1], 'String', 'Optimal
    Consumption Function for \gamma =3', 'EdgeColor', 'none', '
    HorizontalAlignment', 'center','Fontsize',12)
figure
```

```
for i=2:2 %gamma
    for q=1:1 %a
        for k=1:3 %columns of tstar
            t=ones(76,1);
            y=ones(76,1);
            for p=1:76
                t(p)=p+64;
                if (t(p) < tstar(4.*(i-1)+q,k))
                    y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)+q,k))))^(1/gamma(i))*m
                else
                    y(p)=m
                end
            end
            subplot(2,2,1)
            plot(t,y)
            hold all
            legend({'t^*=70','t^*=74','t^*=76'},'FontSize',8)
            title('a=0.10')
            xlabel('t')
            ylabel('c^*(t)')
            axis([65 90 0 inf])
        end
    end
end
```

```
for i=2:2 %gamma
    for q=2:2 %a
        for k=1:3 %columns of tstar
            t=ones(76,1);
            y=ones(76,1);
            for p=1:76
                t(p)=p+64;
                if (t(p) < tstar(4.*(i-1)+q,k))
                    y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
                        exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
                        ))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
                        +q,k))))^(1/gamma(i))*m
                else
                    y(p)=m
                end
            end
            subplot(2,2,2)
            plot(t,y)
            hold all
            legend({'t^*=71','t^*=77','t^*=80'},'FontSize',8)
            title('a=0.05')
            xlabel('t')
            ylabel('c^*(t)')
            axis([65 90 0 inf])
        end
    end
```

```
end

for i=2:2 %gamma
    for q=3:3 %a
        for k=1:3 %columns of tstar
            t=ones(76,1);
            y=ones(76,1);
            for p=1:76
                t(p)=p+64;
                if (t(p) < tstar(4.*(i-1)+q,k))
                    y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
                        exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
                        ))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
                        +q,k))))^(1/gamma(i))*m
                else
                    y(p)=m
                end
            end
            subplot(2,2,3)
            plot(t,y)
            hold all
            legend({'t^*=73','t^*=79','t^*=82'},'FontSize',8)
            title('a=0.03')
            xlabel('t')
            ylabel('c^*(t)')
            axis([65 90 0 inf])
    end
```

```
        end
end

for i=2:2 %gamma
    for q=4:4 %a
        for k=1:3 %columns of tstar
            t=ones(76,1);
            y=ones(76,1);
            for p=1:76
                t(p)=p+64;
                if (t(p) < tstar(4.*(i-1)+q,k))
                    y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)+q,k))))^(1/gamma(i))*m
                else
                    y(p)=m
                end
            end
            subplot(2,2,4)
            plot(t,y)
            hold all
            legend({'t^*=75','t^*=81','t^*=84'},'FontSize',8)
            title('a=0.01')
            xlabel('t')
            ylabel('c^*(t)')
            axis([65 90 0 inf])
```

```
            end
        end
end
annotation('textbox', [0 0.9 1 0.1], 'String', 'Optimal Consumption Function for \gamma =1', 'EdgeColor', 'none', 'HorizontalAlignment', 'center','Fontsize',12)




figure
for i=3:3 %gamma
        for q=1:1 %a
                for k=1:3 %columns of tstar
                        t=ones(76,1);
                        y=ones(76,1);
                        for p=1:76
                                t(p)=p+64;
                                if (t(p) < tstar(4.*(i-1)+q,k))
                                        y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)+q,k))))^(1/gamma(i))*m
                                else
                                        y(p)=m
                                end
                        end
```

```
            subplot(2,2,1)
            plot(t,y)
            hold all
            legend({'t^*=68','t^*=71','t^*=73'},'FontSize',8)
            title('a=0.10')
            xlabel('t')
            ylabel('c^*(t)')
            axis([65 90 0 inf])
        end
    end
end

for i=3:3 %gamma
    for q=2:2 %a
        for k=1:3 %columns of tstar
            t=ones(76,1);
            y=ones(76,1);
            for p=1:76
                t(p)=p+64;
                if (t(p) < tstar(4.*(i-1)+q,k))
                    y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
                        exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
                        ))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
                        +q,k))))^(1/gamma(i))*m
                else
                    y(p)=m
                end
```

```
            end
            subplot(2,2,2)
            plot(t,y)
            hold all
            legend({'t^*=70','t^*=73','t^*=76'},'FontSize',8)
            title('a=0.05')
            xlabel('t')
            ylabel('c^*(t)')
            axis([65 90 0 inf])
        end
    end
end

for i=3:3 %gamma
    for q=3:3 %a
        for k=1:3 %columns of tstar
            t=ones(76,1);
            y=ones(76,1);
            for p=1:76
                t(p)=p+64;
                if (t(p) < tstar(4.*(i-1)+q,k))
                    y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
                        exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
                        ))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
                        +q,k))))^(1/gamma(i))*m
                else
                    y(p)=m
```

```
                end
            end
            subplot(2,2,3)
            plot(t,y)
            hold all
            legend({'t^*=71','t^*=75','t^*=78'},'FontSize',8)
            title('a=0.03')
            xlabel('t')
            ylabel('c^*(t)')
            axis([65 90 0 inf])
        end
    end
end

for i=3:3 %gamma
    for q=4:4 %a
        for k=1:3 %columns of tstar
            t=ones(76,1);
            y=ones(76,1);
            for p=1:76
                t(p)=p+64;
                if (t(p) < tstar(4.*(i-1)+q,k))
                    y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
                        exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
                        ))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
                        +q,k))))^(1/gamma(i))*m
                else
```

```
                y(p)=m
            end
        end
        subplot(2,2,4)
        plot(t,y)
        hold all
        legend({'t^*=73','t^*=78','t^*=80'},'FontSize',8)
        title('a=0.01')
        xlabel('t')
        ylabel('c^*(t)')
        axis([65 90 0 inf])
    end
  end
end
annotation('textbox', [0 0.9 1 0.1], 'String', 'Optimal Consumption Function for \gamma =0.5', 'EdgeColor', 'none', 'HorizontalAlignment', 'center','Fontsize',12)





figure
for i=4:4 %gamma
  for q=1:1 %a
    for k=1:3 %columns of tstar
        t=ones(76,1);
```

```
            y=ones(76,1);
            for p=1:76
                t(p)=p+64;
                if (t(p) < tstar(4.*(i-1)+q,k))
                    y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
                        exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
                        ))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
                        +q,k))))^(1/gamma(i))*m
                else
                    y(p)=m
                end
            end
            subplot(2,2,1)
            plot(t,y)
            hold all
            legend({'t^*=66','t^*=67','t^*=68'},'FontSize',8)
            title('a=0.10')
            xlabel('t')
            ylabel('c^*(t)')
            axis([65 80 0 inf])
        end
    end
end

for i=4:4 %gamma
    for q=2:2 %a
        for k=1:3 %columns of tstar
```

```
            t=ones(76,1);
            y=ones(76,1);
            for p=1:76
                t(p)=p+64;
                if (t(p) < tstar(4.*(i-1)+q,k))
                    y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)+q,k))))^(1/gamma(i))*m
                else
                    y(p)=m
                end
            end
            subplot(2,2,2)
            plot(t,y)
            hold all
            legend({'t^*=67','t^*=69','t^*=70'},'FontSize',8)
            title('a=0.05')
            xlabel('t')
            ylabel('c^*(t)')
            axis([65 80 0 inf])
        end
    end
end

for i=4:4 %gamma
    for q=3:3 %a
```

```
        for k=1:3 %columns of tstar
            t=ones(76,1);
            y=ones(76,1);
            for p=1:76
                t(p)=p+64;
                if (t(p) < tstar(4.*(i-1)+q,k))
                    y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(
                        exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k
                        ))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)
                        +q,k))))^(1/gamma(i))*m
                else
                    y(p)=m
                end
            end
            subplot(2,2,3)
            plot(t,y)
            hold all
            legend({'t^*=68','t^*=70','t^*=71'},'FontSize',8)
            title('a=0.03')
            xlabel('t')
            ylabel('c^*(t)')
            axis([65 80 0 inf])
        end
    end
end

for i=4:4 %gamma
```

```
    for q=4:4 %a
        for k=1:3 %columns of tstar
            t=ones(76,1);
            y=ones(76,1);
            for p=1:76
                t(p)=p+64;
                if (t(p) < tstar(4.*(i-1)+q,k))
                    y(p)=(((exp(-0.00093*(exp(0.087*t(p))-1)))/(exp(-0.00093*(exp(0.087*tstar(4.*(i-1)+q,k))-1))))*exp((j-a(q))*(t(p)-tstar(4.*(i-1)+q,k))))^(1/gamma(i))*m
                else
                    y(p)=m
                end
            end
            subplot(2,2,4)
            plot(t,y)
            hold all
            legend({'t^*=69','t^*=72','t^*=73'},'FontSize',8)
            title('a=0.01')
            xlabel('t')
            ylabel('c^*(t)')
            axis([65 80 0 inf])
        end
    end
end
```

```
annotation('textbox', [0 0.9 1 0.1], 'String', 'Optimal
    Consumption Function for \gamma =0.1', 'EdgeColor', 'none', '
    HorizontalAlignment', 'center','Fontsize',12)
```

2 MATLAB Coed for Accumulation of Wealth Function

```
j=0.03;
gamma=[3 1 0.5 0.1];
a=[0.10 0.05 0.03 0.01];
tstar=[73 80 84; 75 83 88; 77 85 89; 79 87 91; 70 74 76; 71 77
    80; 73 79 82; 75 81 84; 68 71 73; 70 73 76; 71 75 78; 73 78
    80; 66 67 68; 67 69 70; 68 70 71; 69 72 73];



figure
annotation('textbox', [0 0.9 1 0.1], 'String', 'Optimal
    Accumulated Wealth Function for \gamma =3', 'EdgeColor', 'none
    ', 'HorizontalAlignment', 'center','Fontsize',12)
for i=1:1 %gamma
    for q=1:1 %a
        for k=1:3
            m=1;
            fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar
                (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
                (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)))^(-1/gamma(i)
```

```
)).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
).*z)).^(1/gamma(i))))
S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
-1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(
exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
-1)
S00=integral(S0,65,tstar(4*(i-1)+q,k))
t=ones(1,76);
S=ones(1,76);
I=ones(1,76);
y=ones(1,76);
for p=1:76
    t(p)=p+64;
    I(p)=integral(fun,65,t(p))
    S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
    for p=2:76
        if ((S(p)<=0) || (S(p-1)==0) )
            S(p)=0;
        else
            S(p)=S(p);
        end
    end
end
subplot(2,2,1)
plot(t(1:75),S(1:75))
hold all
```

```
                legend({'t^*=73','t^*=80','t^*=84'},'FontSize',8)
                title('\alpha = 0.10')
                xlabel('t')
                ylabel('S^*(t)')
                axis([65 100 0 inf])
            end
        end
end

for i=1:1 %gamma
        for q=2:2 %a
            for k=1:3
                m=1;
                fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)))^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)).*z)).^(1/gamma(i))))
                S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))-1)
                S00=integral(S0,65,tstar(4*(i-1)+q,k))
                t=ones(1,76);
                S=ones(1,76);
                I=ones(1,76);
```

```
                y=ones(1,76);
                for p=1:76
                    t(p)=p+64;
                    I(p)=integral(fun,65,t(p))
                    S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
                    for p=2:76
                        if ((S(p)<=0) || (S(p-1)==0) )
                            S(p)=0;
                        else
                            S(p)=S(p);
                        end
                    end
                end
                subplot(2,2,2)
                plot(t(1:75),S(1:75))
                hold all
                legend({'t^*=75','t^*=83','t^*=88'},'FontSize',8)
                title('\alpha = 0.05')
                xlabel('t')
                ylabel('S^*(t)')
                axis([65 100 0 inf])
            end
        end
end

for i=1:1 %gamma
        for q=3:3 %a
```

```
    for k=1:3
        m=1;
        fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)))^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)).*z)).^(1/gamma(i))))
        S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))-1)
        S00=integral(S0,65,tstar(4*(i-1)+q,k))
        t=ones(1,76);
        S=ones(1,76);
        I=ones(1,76);
        y=ones(1,76);
        for p=1:76
            t(p)=p+64;
            I(p)=integral(fun,65,t(p))
            S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
            for p=2:76
                if ((S(p)<=0) || (S(p-1)==0) )
                    S(p)=0;
                else
                    S(p)=S(p);
                end
```

```
                end
            end
            subplot(2,2,3)
            plot(t(1:75),S(1:75))
            hold all
            legend({'t^*=77','t^*=85','t^*=89'},'FontSize',8)
            title('\alpha = 0.03')
            xlabel('t')
            ylabel('S^*(t)')
            axis([65 100 0 inf])
        end
    end
end

for i=1:1 %gamma
    for q=4:4 %a
        for k=1:3
            m=1;
            fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar
                (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
                (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)))^(-1/gamma(i)
                )).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
                ).*z)).^(1/gamma(i))))
            S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
                -1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
                -1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(
                exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
```

```
            -1)
        S00=integral(S0,65,tstar(4*(i-1)+q,k))
        t=ones(1,76);
        S=ones(1,76);
        I=ones(1,76);
        y=ones(1,76);
        for p=1:76
            t(p)=p+64;
            I(p)=integral(fun,65,t(p))
            S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
            for p=2:76
                if ((S(p)<=0) || (S(p-1)==0) )
                    S(p)=0;
                else
                    S(p)=S(p);
                end
            end
        end
        subplot(2,2,4)
        plot(t(1:75),S(1:75))
        hold all
        legend({'t^*=79','t^*=87','t^*=91'},'FontSize',8)
        title('\alpha = 0.01')
        xlabel('t')
        ylabel('S^*(t)')
        axis([65 100 0 inf])
    end
```

```
        end
    end




figure
annotation('textbox', [0 0.9 1 0.1], 'String', 'Optimal Accumulated Wealth Function for \gamma =1', 'EdgeColor', 'none', 'HorizontalAlignment', 'center','Fontsize',12)
for i=2:2 %gamma
    for q=1:1 %a
        for k=1:3
            m=1;
            fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)))^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)).*z)).^(1/gamma(i))))
            S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))-1)
            S00=integral(S0,65,tstar(4*(i-1)+q,k))
            t=ones(1,76);
            S=ones(1,76);
```

```
            I=ones(1,76);
            y=ones(1,76);
            for p=1:76
                t(p)=p+64;
                I(p)=integral(fun,65,t(p))
                S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
                for p=2:76
                    if ((S(p)<=0) || (S(p-1)==0) )
                        S(p)=0;
                    else
                        S(p)=S(p);
                    end
                end
            end
            subplot(2,2,1)
            plot(t(1:75),S(1:75))
            hold all
            legend({'t^*=70','t^*=74','t^*=76'},'FontSize',8)
            title('\alpha = 0.10')
            xlabel('t')
            ylabel('S^*(t)')
            axis([65 90 0 inf])
        end
    end
end

for i=2:2 %gamma
```

```
    for q=2:2 %a
        for k=1:3
            m=1;
            fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)))^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)).*z)).^(1/gamma(i))))
            S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1)))).^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))-1)
            S00=integral(S0,65,tstar(4*(i-1)+q,k))
            t=ones(1,76);
            S=ones(1,76);
            I=ones(1,76);
            y=ones(1,76);
            for p=1:76
                t(p)=p+64;
                I(p)=integral(fun,65,t(p))
                S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
                for p=2:76
                    if ((S(p)<=0) || (S(p-1)==0) )
                        S(p)=0;
                    else
                        S(p)=S(p);
```

```
                    end
                end
            end
            subplot(2,2,2)
            plot(t(1:75),S(1:75))
            hold all
            legend({'t^*=71','t^*=77','t^*=80'},'FontSize',8)
            title('\alpha = 0.05')
            xlabel('t')
            ylabel('S^*(t)')
            axis([65 90 0 inf])
        end
    end
end

for i=2:2 %gamma
    for q=3:3 %a
        for k=1:3
            m=1;
            fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar
               (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
               (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)))^(-1/gamma(i)
               )).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
               ).*z)).^(1/gamma(i))))
            S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
               -1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
               -1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(
```

```
    exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
    -1)
S00=integral(S0,65,tstar(4*(i-1)+q,k))
t=ones(1,76);
S=ones(1,76);
I=ones(1,76);
y=ones(1,76);
for p=1:76
    t(p)=p+64;
    I(p)=integral(fun,65,t(p))
    S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
    for p=2:76
        if ((S(p)<=0) || (S(p-1)==0) )
            S(p)=0;
        else
            S(p)=S(p);
        end
    end
end
subplot(2,2,3)
plot(t(1:75),S(1:75))
hold all
legend({'t^*=73','t^*=79','t^*=82'},'FontSize',8)
title('\alpha = 0.03')
xlabel('t')
ylabel('S^*(t)')
axis([65 90 0 inf])
```

```
            end
        end
    end

    for i=2:2 %gamma
        for q=4:4 %a
            for k=1:3
                m=1;
                fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)))^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)).*z)).^(1/gamma(i))))
                S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))-1)
                S00=integral(S0,65,tstar(4*(i-1)+q,k))
                t=ones(1,76);
                S=ones(1,76);
                I=ones(1,76);
                y=ones(1,76);
                for p=1:76
                    t(p)=p+64;
                    I(p)=integral(fun,65,t(p))
                    S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
```

```
                    for p=2:76
                        if ((S(p)<=0) || (S(p-1)==0) )
                            S(p)=0;
                        else
                            S(p)=S(p);
                        end
                    end
                end
                subplot(2,2,4)
                plot(t(1:75),S(1:75))
                hold all
                legend({'t^*=75','t^*=81','t^*=84'},'FontSize',8)
                title('\alpha = 0.01')
                xlabel('t')
                ylabel('S^*(t)')
                axis([65 90 0 inf])
            end
        end
end




figure
annotation('textbox', [0 0.9 1 0.1], 'String', 'Optimal
    Accumulated Wealth Function for \gamma =0.5', 'EdgeColor', '
    none', 'HorizontalAlignment', 'center','Fontsize',12)
```

```
for i=3:3 %gamma
    for q=1:1 %a
        for k=1:3
          m=1;
          fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)))^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)).*z)).^(1/gamma(i))))
          S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1)))).^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))-1)
          S00=integral(S0,65,tstar(4*(i-1)+q,k))
          t=ones(1,76);
          S=ones(1,76);
          I=ones(1,76);
          y=ones(1,76);
          for p=1:76
              t(p)=p+64;
              I(p)=integral(fun,65,t(p))
              S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
              for p=2:76
                  if ((S(p)<=0) || (S(p-1)==0) )
                      S(p)=0;
                  else
```

```
                    S(p)=S(p);
                end
            end
        end
        subplot(2,2,1)
        plot(t(1:75),S(1:75))
        hold all
        legend({'t^*=68','t^*=71','t^*=73'},'FontSize',8)
        title('\alpha = 0.10')
        xlabel('t')
        ylabel('S^*(t)')
        axis([65 90 0 inf])
    end
  end
end

for i=3:3 %gamma
  for q=2:2 %a
    for k=1:3
        m=1;
        fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)))^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)).*z)).^(1/gamma(i))))
        S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
```

```
        -1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(
        exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
        -1)
    S00=integral(S0,65,tstar(4*(i-1)+q,k))
    t=ones(1,76);
    S=ones(1,76);
    I=ones(1,76);
    y=ones(1,76);
    for p=1:76
        t(p)=p+64;
        I(p)=integral(fun,65,t(p))
        S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
        for p=2:76
            if ((S(p)<=0) || (S(p-1)==0) )
                S(p)=0;
            else
                S(p)=S(p);
            end
        end
    end
    subplot(2,2,2)
    plot(t(1:75),S(1:75))
    hold all
    legend({'t^*=70','t^*=73','t^*=76'},'FontSize',8)
    title('\alpha = 0.05')
    xlabel('t')
    ylabel('S^*(t)')
```

```
                axis([65 90 0 inf])
            end
        end
end

for i=3:3 %gamma
    for q=3:3 %a
        for k=1:3
            m=1;
            fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)))^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)).*z)).^(1/gamma(i))))
            S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1)))).^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))-1)
            S00=integral(S0,65,tstar(4*(i-1)+q,k))
            t=ones(1,76);
            S=ones(1,76);
            I=ones(1,76);
            y=ones(1,76);
            for p=1:76
                t(p)=p+64;
                I(p)=integral(fun,65,t(p))
```

```
                S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
                for p=2:76
                    if ((S(p)<=0) || (S(p-1)==0) )
                        S(p)=0;
                    else
                        S(p)=S(p);
                    end
                end
            end
            subplot(2,2,3)
            plot(t(1:75),S(1:75))
            hold all
            legend({'t^*=71','t^*=75','t^*=78'},'FontSize',8)
            title('\alpha = 0.03')
            xlabel('t')
            ylabel('S^*(t)')
            axis([65 90 0 inf])
        end
    end
end

for i=3:3 %gamma
    for q=4:4 %a
        for k=1:3
            m=1;
            fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar
                (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
```

```
    (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)))^(-1/gamma(i)
    )).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
    ).*z)).^(1/gamma(i))))
S0=@(x)  exp(-j*(x-65)).*(((exp((j-a(q)).* tstar(4.*(i
    -1)+q,k)).*exp(-0.00093.*(exp((0.087.* tstar(4.*(i
    -1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(
    exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
    -1)
S00=integral(S0,65,tstar(4*(i-1)+q,k))
t=ones(1,76);
S=ones(1,76);
I=ones(1,76);
y=ones(1,76);
for p=1:76
    t(p)=p+64;
    I(p)=integral(fun,65,t(p))
    S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
    for p=2:76
        if ((S(p)<=0) || (S(p-1)==0) )
            S(p)=0;
        else
            S(p)=S(p);
        end
    end
end
subplot(2,2,4)
plot(t(1:75),S(1:75))
```

```
                hold all
                legend({'t^*=73','t^*=78','t^*=80'},'FontSize',8)
                title('\alpha = 0.01')
                xlabel('t')
                ylabel('S^*(t)')
                axis([65 90 0 inf])
            end
        end
end




figure
annotation('textbox', [0 0.9 1 0.1], 'String', 'Optimal Accumulated Wealth Function for \gamma =0.1', 'EdgeColor', 'none', 'HorizontalAlignment', 'center','Fontsize',12)
for i=4:4 %gamma
    for q=1:1 %a
        for k=1:3
            m=1;
            fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)))^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)).*z)).^(1/gamma(i))))
            S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
```

```
    -1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
    -1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(
    exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
    -1)
S00=integral(S0,65,tstar(4*(i-1)+q,k))
t=ones(1,76);
S=ones(1,76);
I=ones(1,76);
y=ones(1,76);
for p=1:76
    t(p)=p+64;
    I(p)=integral(fun,65,t(p))
    S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
    for p=2:76
        if ((S(p)<=0) || (S(p-1)==0) )
            S(p)=0;
        else
            S(p)=S(p);
        end
    end
end
subplot(2,2,1)
plot(t(1:75),S(1:75))
hold all
legend({'t^*=66','t^*=67','t^*=68'},'FontSize',8)
title('\alpha = 0.10')
xlabel('t')
```

```
            ylabel('S^*(t)')
            axis([65 80 0 inf])
        end
    end
end

for i=4:4 %gamma
    for q=2:2 %a
        for k=1:3
            m=1;
            fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar
                (4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
                (4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)))^(-1/gamma(i)
                )).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
                ).*z)).^(1/gamma(i))))
            S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
                -1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
                -1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(
                exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
                -1)
            S00=integral(S0,65,tstar(4*(i-1)+q,k))
            t=ones(1,76);
            S=ones(1,76);
            I=ones(1,76);
            y=ones(1,76);
            for p=1:76
                t(p)=p+64;
```

```
                    I(p)=integral(fun,65,t(p))
                    S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
                    for p=2:76
                        if ((S(p)<=0) || (S(p-1)==0) )
                            S(p)=0;
                        else
                            S(p)=S(p);
                        end
                    end
                end
                subplot(2,2,2)
                plot(t(1:75),S(1:75))
                hold all
                legend({'t^*=67','t^*=69','t^*=70'},'FontSize',8)
                title('\alpha = 0.05')
                xlabel('t')
                ylabel('S^*(t)')
                axis([65 80 0 inf])
            end
        end
    end

for i=4:4 %gamma
    for q=3:3 %a
        for k=1:3
            m=1;
            fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar
```

```
(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar
(4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)))^(-1/gamma(i)
)).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)
).*z)).^(1/gamma(i))))
S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i
-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i
-1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(
exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))
-1)
S00=integral(S0,65,tstar(4*(i-1)+q,k))
t=ones(1,76);
S=ones(1,76);
I=ones(1,76);
y=ones(1,76);
for p=1:76
    t(p)=p+64;
    I(p)=integral(fun,65,t(p))
    S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
    for p=2:76
        if ((S(p)<=0) || (S(p-1)==0) )
            S(p)=0;
        else
            S(p)=S(p);
        end
    end
end
subplot(2,2,3)
```

```
                plot(t(1:75),S(1:75))
                hold all
                legend({'t^*=68','t^*=70','t^*=71'},'FontSize',8)
                title('\alpha = 0.03')
                xlabel('t')
                ylabel('S^*(t)')
                axis([65 80 0 inf])
            end
        end
end

for i=4:4 %gamma
        for q=4:4 %a
            for k=1:3
                m=1;
                fun=@(z) (exp(-j.*(z-65))).*(1-(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1)).*m.^(-gamma(i)))^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*z)-1)).*exp((j-a(q)).*z)).^(1/gamma(i))))
                S0=@(x) exp(-j*(x-65)).*(((exp((j-a(q)).*tstar(4.*(i-1)+q,k)).*exp(-0.00093.*(exp((0.087.*tstar(4.*(i-1)+q,k)))-1))).^(-1/gamma(i))).*(exp(-0.00093.*(exp(0.087.*x)-1)).*exp((j-a(q)).*x)).^(1/gamma(i))-1)
                S00=integral(S0,65,tstar(4*(i-1)+q,k))
                t=ones(1,76);
```

```
                S=ones(1,76);
                I=ones(1,76);
                y=ones(1,76);
                for p=1:76
                    t(p)=p+64;
                    I(p)=integral(fun,65,t(p))
                    S(p)=exp(j*(t(p)-65))*(S00*m+I(p))
                    for p=2:76
                        if ((S(p)<=0) || (S(p-1)==0) )
                            S(p)=0;
                        else
                            S(p)=S(p);
                        end
                    end
                end
                subplot(2,2,4)
                plot(t(1:75),S(1:75))
                hold all
                legend({'t^*=69','t^*=72','t^*=73'},'FontSize',8)
                title('\alpha = 0.01')
                xlabel('t')
                ylabel('S^*(t)')
                axis([65 80 0 inf])
            end
        end
    end
```